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BY

LLOYD WILLIAM TAYLOR

INSTRUCTOR IN PHYSICS IN THE UNIVERSITY OF CHICAGO

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PREFACE

The exhaustion of the printed supply of Mann's "Manual of Advanced Optics," and the concurrence of its author in an opinion that a radical revision was necessary, have furnished the occasion for the present volume. During the growth of this manual the primary aim has been the coördination of laboratory and class work. It is neither a laboratory manual, in the ordinary sense of the term, nor is it simply a classroom text. It is intended to take the place of both.

The presentation of sciences in water-tight compartments labeled "Laboratory Courses" and "Class Courses" is perhaps necessary from an administrative point of view. Unquestionably, however, it erects artificial barriers between closely related lines of thought, and thereby introduces troublesome and unnecessary difficulties in coördination. This manual is intended to bridge the gap between these artificially separated portions of the study of light.

Certain portions of this volume, particularly the laboratory instructions, may perhaps be charged with being dogmatic. They *are* dogmatic — purposely so. When a student enters an unfamiliar field of laboratory manipulation, the simplest and most concise instructions are none too easily grasped. The enumeration of several alternative methods of carrying out an experiment, with comments, admonitions, and cautions about each, can do nothing else than throw him into inextricable confusion. If one sends him peremptorily on one route, not mentioning permissible deviations or equally good alternative ways, the chance is much greater that he will reach his destination. It is only after he has once arrived that he can profitably consider other paths to the same place.

In drawing up the material for this text the author has unhesitatingly drawn from any source which seemed capable of contributing to the plan in view. At the same time, he has so modified the material chosen as to fit the text to the assumed capacities of prospective students. For the general plan of the text, credit is due his predecessor, C. R. Mann. For the suggestion which resulted in its growth, as well as for valuable advice concerning material and manner of presentation, the author tenders grateful acknowledgment to Professor H. G. Gale. The treatment of lens theory closely follows the first portion of Drude's "Theory of Optics." The diffraction problem receives much the same treatment as is found in Ollivier's "Cours de Physique Générale." To Professor A. A. Michelson and The University of Chicago Press, acknowledgment is made of permission to use material from "Light Waves and their Uses." The author also takes pleasure in expressing his indebtedness to Professor R. A. Millikan for permission to make extensive quotations from Chapter XXVII of "Electricity, Sound, and Light" and to Professor D. C. Miller for similar permission in connection with Chapter XV of "Laboratory Physics." Professor Miller has, moreover, generously accorded permission to use cuts for Figs. 37, 58, 59, 104, 104 A, 105, 106, and 107 of this text, in addition to those accompanying the material mentioned above. Professor H. B. Lemon has given active coöperation in drawing up the material for Chapter XV. Mr. Fred Pearson has constructively criticized the directions for silvering mirrors.

To Dr. F. G. Tucker and Mrs. G. S. Monk the author is indebted for a careful reading of the manuscript and for a number of suggestions which have contributed to clarity of statement. Notwithstanding their careful inspection, there will undoubtedly be errors in the final impression, as always in a publication of this type. Information leading to correction of these will be gratefully received.

LLOYD W. TAYLOR

THE UNIVERSITY OF CHICAGO

CONTENTS

PAGE

CHAPTER I. OBJECT AND IMAGE 1

Geometrical and physical optics, 1. Cardinal points of an optical system, 2. Fundamental equations of an optical system, 3. Principal foci, 5. Lateral magnification, 6. Principal planes and points, 6. Use of principal planes and foci in graphical constructions, 6. Equations (5) referred to principal planes, 7. Angular magnification, 8. Nodal points, 8. Use of nodal points in graphical constructions, 9. Experimental location of nodal points, 9. Relation of nodal points to principal points, 11. Combination of two systems, 11. The power of an optical system, 14. Combinations of thin lenses, 14. Thin lenses in contact, 16.

EXPERIMENT I. CARDINAL POINTS OF A TWO-COMPONENT LENS SYSTEM 18

CHAPTER II. THE LENS 21

Lens Laws: Usual type of optical system, 21. Refraction at a single spherical surface, 21. Ray-tracing through optical systems, 22. Lateral magnification, 23. Principal planes, 23. Nodal points, 23. Principal focal lengths, 24. Lenses, 24. Thin lenses, 26. *Aberrations:* Meaning of the term *aberration*, 28. Types of aberration, 28. Correction of aberrations, 31.

EXPERIMENT II. *A*. INDEX OF REFRACTION OF A CONVERGING LENS 31

B. THE ABERRATIONS OF A SIMPLE LENS 33

CHAPTER III. THE SINGLE SLIT 36

Meaning of *diffraction*, 36. Pattern due to a single slit, 36. Intensity function for single slit, 37. Minima and maxima of intensity, 39. Resolving power of a single slit, 41.

EXPERIMENT III. WAVE-LENGTH BY SINGLE-SLIT DIFFRACTION PATTERN 42

CHAPTER IV. THE DOUBLE SLIT 44

Pattern due to two slits, 44. Applications, 44. Intensity function for double slit, 45. Maxima and minima of intensity, 45. Missing orders, 48. Resolving power of a double slit, 48. Application to measurement of stellar magnitudes, 49.

PAGE

EXPERIMENT IV. THE ANGULAR DIAMETER OF A DISTANT SOURCE 50

CHAPTER V. *N* SLITS 53

Pattern due to *N* slits, 53. The diffraction grating, 53. Intensity function for *N* slits, 54. Minima and maxima of intensity, 56. Minimum deviation, 57. Width of principal maxima, 58. Dispersion, 59. Normal spectra, 59. Overlapping of spectra, 60. Resolving power of a grating, 60. Relation between dispersion and resolving power, 61. Simple theory of slit and grating, 62.

EXPERIMENT V. THE DIFFRACTION GRATING 62

CHAPTER VI. THE PRISM 67

Nature and function of prisms, 67. Course of a ray through a principal section, 67. Minimum deviation, 69. Index of refraction, 70. Dispersion, 70. Thickness of prism, 72. Resolving power, 73. Relation between dispersion and resolving power, 74. Determination of Cauchy constants, 75. Description of dispersive power, 75. The direct-vision spectroscope, 76.

EXPERIMENT VI. *A*. MEASUREMENT OF ANGLE OF PRISM . . . 77

B. INDEX OF REFRACTION BY METHOD OF MINIMUM DEVIATION 80

C. DISPERSION CURVE OF PRISM 81

D. MEASUREMENT OF RESOLVING POWER 82

CHAPTER VII. THE CRITICAL ANGLE 84

Critical angle of refraction, 84. Index of refraction by total reflection, 85. Index of refraction by grazing incidence, 86. The refractometer, 87. Index of Abbe prism, 88.

EXPERIMENT VII. INDEX OF REFRACTION BY GRAZING INCIDENCE 88

A. INDEX OF A PRISM 88

B. INDEX OF REFRACTION OF A LIQUID 89

C. INDEX OF GLASS PLATE DETERMINED BY ABBE PRISM . . 90

CHAPTER VIII. INTERFERENCE BETWEEN TWO BEAMS . . 91

Conditions necessary for interference, 91. Discussion of conditions, 91. Methods of producing interference, 94. The Fresnel mirrors, 94. Measurement of wave-length by the Fresnel mirrors, 96. The Fresnel bi-prism, 97. Measurement of wave-length by Fresnel bi-prism (divergent light), 97. Measurement of wave-length by Fresnel bi-prism (parallel light), 99. White-light fringes, 99.

EXPERIMENT VIII. *A*. MEASUREMENT OF WAVE-LENGTH BY A FRESNEL BI-PRISM 99

B. MEASUREMENT OF WAVE-LENGTH BY THE FRESNEL MIRRORS 103

PAGE

CHAPTER IX. THE MICHELSON INTERFEROMETER 107

Limitations on interference applications, 107. The interferometer, 108. Production of interference fringes, 110. Position of fringes, 113. Form of fringes, 113. Measurement of distance and wave-length by the interferometer, 115. Measurement of index of refraction by the interferometer, 116. The interferometer in spectroscopy, 117. Visibility curves, 118.

EXPERIMENT IX. *A*. CALIBRATION OF A TENTH-MILLIMETER SCALE BY THE INTERFEROMETER 120

B. MEASUREMENT OF REFRACTIVE INDEX OF A GLASS PLATE BY THE INTERFEROMETER 125

CHAPTER X. THE PARALLEL-PLATE INTERFEROMETER . . 129

Interference in multiple beams, 129. The parallel-plate interferometer, 129. Nature of the fringes, 130. The intensity function, 131. Maxima and minima, 133. Width of maxima, 133. Relation of intensity of minima to maxima, 134. Form of fringes, 135.

EXPERIMENT X. RATIO OF WAVE-LENGTHS OF YELLOW LINES OF MERCURY 135

CHAPTER XI. POLARIZED LIGHT 138

Polarization by reflection, 138. Polarization by double refraction, 143. Theory of double refraction, 147. Construction of the Nicol prism, 149. Construction of the Wollaston prism, 149. Passage of polarized light through thin crystals, 150. Double refraction produced by strain, 157. Polarized-light effects in crystals cut perpendicular to axis, 158. Rotation of the plane of polarization, 160.

CHAPTER XII. ROTATION OF THE PLANE OF POLARIZATION 163

Optical activity, 163. Specific rotation, 164. Sugar-testing, 165. Nature of inversion process, 166.

EXPERIMENT XII. PURITY OF SUGAR TESTED BY POLARIMETER 167

CHAPTER XIII. ELLIPTICALLY POLARIZED LIGHT 174

Production and measurement, 174. Equation of an elliptical vibration, 175. Properties of an elliptical vibration, 176. Method of measurement, 177.

EXPERIMENT XIII. ANALYSIS OF ELLIPTICALLY POLARIZED LIGHT 179

CHAPTER XIV. THE REFLECTION OF POLARIZED LIGHT . . 182

Directions of reflected and of refracted light, 182. Intensities of reflected and of refracted light, 182. Bearing of Fresnel's equations upon light theories, 183. Consequences of Fresnel's equations, 184. Experimental verification of Fresnel's equations, 186. Rotation of plane of polarization by reflection, 186.

PAGE

Experiment XIV. Verification of Fresnel's Laws of Reflection 187

CHAPTER XV. THE SPECTROPHOTOMETER 190

Comparison of luminous intensities, 190. The spectrophotometer, 190. The Brace-Lemon spectrophotometer, 191. The Brace prism, 191. Comparison of intensities, 192.

Experiment XV. Comparison of Spectral Distributions of Intensities of Various Light Sources 193

APPENDIX I. ILLUMINATION 197

APPENDIX II. LABORATORY MANIPULATIONS 201

The silvering of glass surfaces, 201. The adjustment of the spectrometer, 207. The use of the vernier scale, 211. The use of micrometer calipers, 212. The use of the spherometer, 213. The use of the micrometer microscope, 214. The use of the analytical balance, 215. Adjustment and use of the telescope and the scale for angle measurement, 216. The Babinet compensator, 217.

APPENDIX III. TABLES 220

Sin² θ for spectrophotometer, 220, 221. Natural and logarithmic sines, 222–225. Natural and logarithmic tangents, 226–229. Logarithms, 230, 231.

INDEX 233

INTRODUCTORY NOTE

The last five years have witnessed a remarkable growth of interest in experimental optics. Part of this is a consequence of notable advances in certain fields of astronomical measurement. More largely, however, it is due to the contributions which spectroscopic research is making to our knowledge of the structure of the atom.

The introduction of the elementary physics student to this field has been unnecessarily difficult in the past by reason of the scarcity of suitable texts. In most of those which have existed, the descriptive and theoretical material has almost or quite entirely crowded out the experimental. The exhaustion of the printed supply of the manual which has been in use in The University of Chicago for the last twenty years, and which has been one of the very few appropriate for use in the laboratory, has furnished a welcome occasion for introducing some rather essential modifications.

In facilitating the introduction of the beginner to this powerful field of research, the author of the present volume is filling a definite need in the development of this field of physics.

HENRY G. GALE

THE UNIVERSITY OF CHICAGO

COLLEGE MANUAL OF OPTICS

CHAPTER I

OBJECT AND IMAGE

Geometrical and physical optics. Statements are sometimes observed in elementary textbooks on physics to the effect that, except as influenced by reflection and refraction, light is propagated in straight lines. It may be shown by simple experiments that this is not in general true, though the deviation from rectilinear propagation is usually small.

If, taking advantage of the minuteness of these deviations, we simply neglect them, and in addition to thus assuming rectilinear propagation we assume also the familiar " laws " of reflection and refraction, we have the foundation for what is known as *geometrical optics.* Many optical phenomena, among them those which have received the most extensive practical application, may be dealt with on the basis of these assumptions.

On the other hand, we may postulate not rectilinear propagation but an undulatory nature of light, one consequence of which is a departure from rectilinear propagation. The deduction of the consequences of this assumption forms a large part of a field of study termed *physical optics.*

These two fields, geometrical and physical optics, have about as little in common as could be imagined in two subjects dealing with the same branch of physics. The former consists largely of the establishment of a basis for the design of optical instruments, whereas the latter is concerned mainly with the nature of light. In many texts on optics one field is treated to the entire exclusion of the other.

The point of view of this text is primarily that of physical optics. Contrary to the general plan, however, the first two chapters will adopt the point of view of geometrical optics.

In what follows, the student will be assumed to be familiar with the elementary concepts of optics. A working knowledge of them renders unnecessary the definition of such terms as *light ray*, *mirror*, *lens*, *object*, *real* and *virtual images*, *principal* and *conjugate foci*, etc. If such familiarity does not exist, any good text on college physics will supply the deficiency; for example, Millikan and Mills' "Electricity, Sound, and Light."

Cardinal points of an optical system. The student will recall that, in making and interpreting calculations on mirrors, there was seldom any doubt as to what should be considered the reference point of a mirror when specifying distance; for example, from a concave mirror to its principal focus. It will be readily seen that there is no such definiteness in a statement concerning the principal focal length of a lens, and still less for a combination of lenses. For such systems possess finite, sometimes great, thickness; and it is necessary to specify what point in the system is to be regarded as the reference point.

In some cases, for example, spectacle lenses, the thickness of the lens is so minute in comparison with its focal length that for practical purposes it makes no difference what point in the lens is taken as the reference point. Such lenses are termed *thin*.

For any other than thin lenses, however, the difficulty is not so easily surmounted. For these it is found convenient to choose two points, which have become known as *principal points*, the distances from which to the principal foci are defined as the focal lengths. It is found convenient for some purposes to define also two other points, which are termed *nodal points*. The three pairs of points (principal foci, principal points, and nodal points) all lie on the optical axis of the lens or lens system to which they belong, and are known collectively as the *cardinal points* or *Gauss points*. They play a very important part in the development of geometrical optics.

Fundamental equations of an optical system. In the following paragraphs some of the properties of the cardinal points are deduced. It is interesting to note in passing that, in the equations expressing the chief facts pertaining to image formation, no assumptions are necessary concerning the optical or physical properties of the system which brings such formation about, nor concerning the nature of light. Only the existence of an object and unique image is assumed. The theory proceeds from the assumption that for every point, line, and plane in a so-called *object space* there exists one and only one point, line, or plane, respectively, in the *image space*, and deduces the consequences of this assumption. The failure of optical systems to fulfill completely these conditions leads to imperfections in the image which are called *aberrations*. These may be dealt with independently of the theory just referred to.

Following a method often given the name of *Abbe's theory*, let us assume that by any means whatever (for example, by a lens) all the points of a given object space are imaged in points of image space. These sets of points may be located by two systems of coördinates, which we shall call x, y, z for the object space and x', y', z' for the image space.

Considerations of projective geometry into which it is not profitable to enter here, show us that if every point, line, or plane in the object space is imaged as a single point, line, or plane in the image space, the following relations exist between these two systems of coördinates ("collinear" relation).

$$\left.\begin{aligned} x' &= \frac{a_1x + b_1y + c_1z + d_1}{ax + by + cz + d}, \\ y' &= \frac{a_2x + b_2y + c_2z + d_2}{ax + by + cz + d}, \\ z' &= \frac{a_3x + b_3y + c_3z + d_3}{ax + by + cz + d}. \end{aligned}\right\} \qquad (1)$$

Equations (1) may be simplified by imposing some simple conditions on the orientation of the axes and position of the origin, without in any way affecting their generality. To effect this simplification let us confine attention to one of the planes passing through the axis of the lens system, and let this be the plane of the paper (Fig. 1).

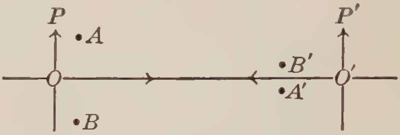

FIG. 1. Choice of axes and directions

Assume the object and image space coaxial, the common axis being the x coördinate axis, the positive direction of x being to the right and that of x' to the left. Let y and y' be perpendicular to the xx' axis, positive upward. The z and z' coördinates no longer appear in the equations (1), which therefore assume the form

$$\left.\begin{aligned} x' &= \frac{a_1x + b_1y + d_1}{ax + by + d}, \\ y' &= \frac{a_2x + b_2y + d_2}{ax + by + d}. \end{aligned}\right\} \tag{2}$$

If we now assume that we are dealing with a symmetrical optical system, such that the images A' and B' possess the same abscissa in the image space whenever the point objects A and B possess the same abscissa in the object space, then when y changes sign the value of x' remains unaltered, and the only effect on y' is a corresponding change of sign. These two conditions are fulfilled in equations (2) only when $b = b_1 = 0$ and $a_2 = d_2 = 0$, whence equations (2) reduce to

$$\left.\begin{aligned} x' &= \frac{a_1x + d_1}{ax + d}, \\ y' &= \frac{b_2y}{ax + d}. \end{aligned}\right\} \tag{3}$$

Solving (3) for x and y gives

$$\left.\begin{aligned} x &= \frac{d_1 - dx'}{ax' - a_1}, \\ y &= \frac{y'(a_1d - ad_1)}{b_2(a_1 - ax')}. \end{aligned}\right\} \tag{3′}$$

Principal foci. As in the case of spherical mirrors, the principal focus may be defined as the point at which exists the real or virtual image of an infinitely distant object. The condition for an infinitely distant object in equations (3′) is $x=\infty$, which can only be satisfied if $ax'-a_1=0$.

In contradistinction to a spherical mirror, a lens possesses *two* principal foci, and the second is readily seen to be given by the condition on (3) that $x'=\infty$; whence $ax+d=0$.

If now, for the purpose of further simplification, we locate the origins of the two sets of coördinates at the appropriate principal foci, (3) and (3′) must take such a form that x' becomes infinite when $x=0$ and x becomes infinite when $x'=0$. As equations (3) and (3′) now stand, when $x'=\infty$, $x=-\frac{d}{a}$, and when $x=\infty$, $x'=\frac{a_1}{a}$. We must therefore shift the origin of x to the left by an amount $\frac{d}{a}$, and that of x' to the left by an amount $\frac{a_1}{a}$. These transformations may be effected by the familiar method of analytic geometry; the first by the substitution $x_0=x+\frac{d}{a}$ and the second by the substitution $x_0'=x'-\frac{a_1}{a}$, where x_0 and x_0' are abscissas in the transformed coördinate systems. Substitution in either (3) or (3′) of the values of x and x' in terms of x_0 and x_0' gives the equations

$$\left.\begin{aligned} x_0x_0' &= \frac{ad_1-a_1d}{a^2}, \\ \frac{y'}{y} &= \frac{b_2}{ax_0}. \end{aligned}\right\} \tag{4}$$

For the purpose of further simplification we shall introduce two new constants, f and f', defined by the relations

$$\frac{ad_1-a_1d}{a^2}=ff',$$

$$\frac{b_2}{a}=f.$$

Substituting these into (4) and dropping the zero subscripts, there result

$$\left.\begin{aligned} xx' &= ff', \\ \frac{y'}{y} &= \frac{f}{x}\left(=\frac{x'}{f'}\right). \end{aligned}\right\} \qquad (5)$$

The student is expected to verify the parenthesized part of the last equation of (5).

Lateral magnification. The fraction $\frac{y'}{y}$ expresses the ratio of the distance from the axis to a given ray at a predetermined abscissa in the image space to the corresponding distance at the appropriate abscissa in the object space. This ratio is commonly known as the *lateral magnification* (β). It is evident that for $x=f$ or $x'=f'$ the lateral magnification is unity.

Principal planes and points. The ratio $\beta=1$ exists for only one pair of points on the ray considered, and hence may be regarded as the defining condition for two planes. The two planes perpendicular to the axis at which the ratio β is unity are known as the *principal planes*, and their intersections with the axis as the *principal points*. Hence, if an object were placed in one of the principal planes, there would exist in the other principal plane an image, real or virtual, having the same size as the object, and erect. For this reason the principal planes are sometimes called *unit planes* and the principal points *unit points*.

We may consider that there have been defined two points on the axis, one of them a distance f from the principal focus of the object space, and the other a distance f' from the principal focus of the image space. Or we may reverse the point of view and consider that we now have two points from which we may measure the principal focal distances. In other words, the dilemma indicated on page 2 has been solved, and *the principal points are the points from which the principal focal distances f and f' are to be measured.*

Use of principal planes and foci in graphical constructions. The location of the image of any point not on the axis may be readily found by utilizing the properties of the principal planes (AB and $A'B'$) and foci (F and F') in accordance with Fig. 2.

It is of interest to note that the path (real or virtual) of the ray $OAA'O'$ between the points A and A' is completely immaterial. In a simple lens it will be fairly direct. In a complicated lens system it may be exceedingly devious. The important point is that every ray directed toward A in the object space has such a direction in the image space that it appears to have come from A'.

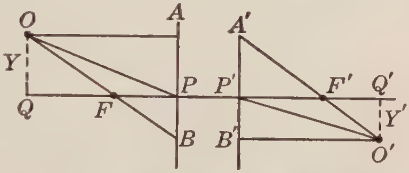

FIG. 2. Properties of principal planes

Equations (5) referred to principal planes. For some purposes it is more convenient to measure axial distances from the principal planes instead of the principal focal planes. Equations (5) take the following form when so expressed:

$$\left.\begin{aligned}\frac{f}{a}+\frac{f'}{a'}&=1\\ \frac{y'}{y}&=\frac{fa'}{f'a}\end{aligned}\right\}, \qquad (5')$$

where a and a' are the new abscissas referred to the principal points as origins, a being positive to the left and a' positive to the right.

These relations are derived from equations (5) by the simple expedient of again shifting the origins. In Fig. 2 this shift is to the right (from F to P) in the object space, and to the left (from F' to P') in the image space. The corresponding transformation equations are therefore $a = x + f$ and $a' = x' + f'$. Substituting in the first of equations (5) the values of x and x' in terms of the new abscissas a and a',

$$(f+a)(f'+a')=ff';$$

whence

$$\frac{f}{a}+\frac{f'}{a'}=1.$$

The last equation may be written $f=\frac{a}{a'}(a'-f')$. Eliminating f between this equation and the equation $x=a-f$ gives $x=\frac{af'}{a'}$. The substitution of this value for x in $\frac{y'}{y}=\frac{f}{x}$ gives

$$\frac{y'}{y}=\frac{fa'}{f'a}. \qquad \text{Q.E.D.}$$

Angular magnification. Let QA and $A'Q'$ (Fig. 3) be conjugate rays of an optical system. Let them make angles u and u' respectively with the axes.

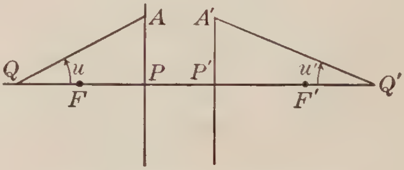

FIG. 3. Angular magnification

The ratio $\frac{\tan u'}{\tan u}$ is called the *angular magnification* (γ), or *convergence ratio*, of the system. From the figure, $\tan u=\frac{AP}{QP}=\frac{AP}{f-x}$ (since F is taken as the origin of the object space with its positive direction to the right; see page 4). Similarly, $\tan u'=\frac{A'P'}{Q'P'}=\frac{A'P'}{f'-x'}$. Noting that $AP=A'P'$ (being in the principal plane) and dividing,

$$\gamma=\frac{\tan u'}{\tan u}=\frac{a}{a'}=\frac{f-x}{f'-x'}=-\frac{x}{f'}=-\frac{f}{x'}, \tag{6}$$

where

$$a=QP,$$
$$a'=Q'P'.$$

The student is expected to prove the correctness of the last two members of equation (6).

Nodal points. We have thus an expression for the angular magnification in terms of the position of the object and image distances, and the principal focal lengths. If in (6) $x=f'$, or $x'=f$, the angular magnification assumes the value -1. These values of the abscissas thus define two points on the axis for which the incident and emergent rays are parallel and conjugate. The points thus defined are the *nodal points*.

Use of nodal points in graphical constructions. Except for a very important connection with the principal points (a connection which will be explained presently), the nodal points are not very important from the standpoint of practice in graphical construction. Their chief geometric property will be evident from the diagram of Fig. 4.

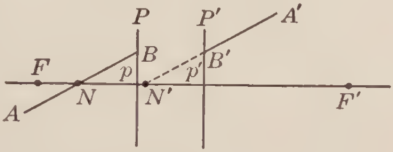

FIG. 4. Chief property of nodal points

The ray AB, passing through N (the incident nodal point), will have $B'A'$ for its conjugate ray, parallel to AB. Also, since pP and $p'P'$ represent the principal planes, $pB = p'B'$. If $B'A'$ be produced backward until it cuts the axis, the point thus determined (N') is the emergent nodal point.

Experimental location of nodal points. The usual process of experimentally locating the nodal points of an optical system utilizes the fact that, if light from a source at infinity is incident at any angle, the position of the image on a screen which is perpendicular to the direction of the incident light is unaffected by rotation of the system about a line perpendicular to the principal axis and passing through the emergent nodal point. The panoramic camera takes advantage of this fact. The existence of this property of the nodal points will be evident from the following considerations.

A straight line (for example, the emergent ray now being considered) is uniquely determined by two conditions. These may be that it shall pass through a given point and have a given direction. Now the direction of the emergent beam which apparently passes through the emergent nodal point is fixed by the direction of the incident parallel light. The only factor which can affect it is a change in the direction of this incident light. No rotation of the optical system about any axis whatever will cause a change of direction.

If, now, such a rotation should occur about an axis perpendicular to the optical axis and passing through the emergent nodal point, it is plain that the position of that point would remain unchanged, and consequently the position of any line of fixed direction passing through it would also be unaffected.

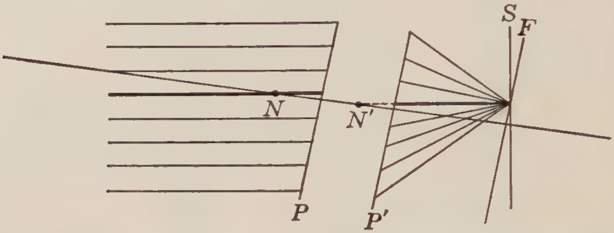

Fig. 5. Experimental determination of nodal points (first position)

This would not be the case for a rotation of the optical system about any axis other than one passing through the emergent nodal point, for in this case the emergent ray, though remaining undeflected, would suffer a lateral displacement.

Examination of Figs. 5 and 6 will show that the rotation described, while it would not cause any *displacement* of the image, would cause it to go slightly in and out of focus. For if the focal plane and the plane of the screen should intersect at the location of the image at the position of Fig. 5, they would not intersect there after the optical system had been rotated to the position of Fig. 6. The image in the latter case would be in

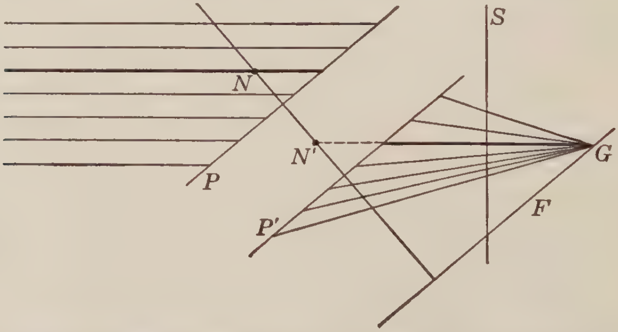

Fig. 6. Experimental determination of nodal points (second position)

focus at the point G and not at the point where the emergent ray strikes the screen. Curvature of the field (see page 30) tends to counteract this.

Relation of nodal points to principal points. Reference to Fig. 4 will show, by comparison of identical triangles, that since $Np = N'p'$, then $NN' = pp'$; that is, the distance between nodal points is the same as that between principal points.

Moreover, equation (6) shows that when, at the same time, $x = f'$ and $f = f'$, then $x = x' = f$, and hence the nodal points coincide with the principal points under this condition. It may be shown (p. 24) that the ratio of the focal lengths of an optical system depends only on the ratio of the indices of refraction of the image and object spaces and has the value *1* when these indices are the same. Hence it follows that, for all cases in which the indices of the initial and final media are the same, *the nodal points are identical with the principal points.* This condition is the one which almost always holds in practice, and consequently as a rule the determination of the nodal points by the method described above locates also the principal points.

Combination of two systems. In the foregoing no limitation has been placed on the optical system discussed, except that there has been postulated its ability to produce an image of a given object, point for point, line for line, and plane for plane. The system could have been composed of a simple lens or mirror, or a combination of several. In either case the same analysis was applicable. The fact that any combination, however complex, may be treated as a single optical unit is of very great value.

There often occurs the necessity for finding the cardinal points of the single system, which may be regarded as the equivalent of the combination formed by a series of two or more systems. We shall develop the equations connecting the characteristics of the combination with those of its components for a two-component system. The combination of a more complex system may be effected by successive applications of the two-component results.

In Fig. 7 let the subscript *1* refer to the first component; *2*, to the second; and the letters not bearing a subscript, to the combination. Suppose the distance $F_1'F_2$ to be known. This is termed the separation. Call it Δ, and let it be positive if F_2 lies to the right of F_1'. Expressions are required for the distances $F'F_2$ and FF_1 (the latter not shown) and for the focal lengths f' and f (the latter not shown) in terms of the known constants of the component systems.

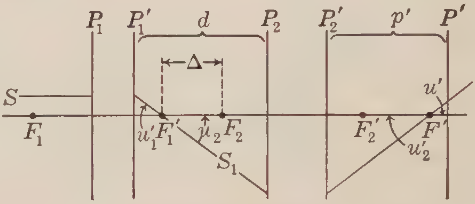

FIG. 7. Combinations of two systems

A ray S, initially parallel to the axis, will be transformed by the first component into S_1, passing through the principal focus of that component. S_1 will be transformed by the second component into S'. The point of intersection of S' with the axis is the principal focus of the combination, and may be found by utilizing the first of equations (5) as follows:

$$F_2'F' = \frac{f_2 f_2'}{\Delta} \quad \text{and} \quad FF_1 = \frac{f_1 f_1'}{\Delta}. \tag{7}$$

The second follows by symmetry from the first, or the figure may be relettered, interchanging object and image space, and the second relation thus verified. $F'F_2'$ is to be taken positive if F' lies to the right of F_2', and FF_1 positive if F lies to the left of F_1. The position of the emergent principal plane P' of the combination is determined by the intersection of the ray S' with S produced. Why?

To find the principal focal lengths of the combination apply equation (6) to the second component (rays S_1 and S'):

$$\frac{\tan u_2'}{\tan u_2} = -\frac{x_2}{f_2'}.$$

Now $x_2 = -\Delta$, $\tan u_2' = \tan u' = \frac{y}{f'}$, $\tan u_2 = \tan u_1' = \frac{y}{f_1'}$. Substituting these values in the above expression,

$$\frac{\frac{y}{f'}}{\frac{y}{f_1'}} = -\frac{\Delta}{f_2'},$$

or
$$f' = -\frac{f_1' f_2'}{\Delta} \quad \text{and} \quad f = -\frac{f_1 f_2}{\Delta}. \tag{8}$$

The equation for f follows that for f' by symmetry. Δ is to be considered positive when the adjacent principal focal distances do not overlap.

Equations (7) give the *positions of the principal foci* of the combination with reference to the principal foci of the components. Equations (8) give the *principal focal lengths* of the combination, and hence locate the *principal planes* of the combination with reference to the principal foci.

It is sometimes convenient, however, to know the distance p' between the second principal plane of the combination and that of the second component, and the corresponding distance p between the first principal plane of the combination and that of the first component.

From Fig. 7 $\quad p' = P_2'P' = f_2' + F_2'F' + f'$.

But from the first of equations (7)

$$F_2'F' = \frac{f_2 f_2'}{\Delta},$$

and from the first of equations (8)

$$f' = -\frac{f_1' f_2'}{\Delta}.$$

Substituting these in the expression for p' there results

$$p' = \frac{f_2'(f_1' + \Delta + f_2)}{\Delta}, \text{ and, by symmetry, } p = \frac{f_1(f_1' + \Delta + f_2)}{\Delta}, \tag{9}$$

which gives p' and p in terms of known constants.

It should be noted that the expression in parentheses in each of equations (9) represents the distance (d) between adjacent principal planes of the component systems. It is sometimes convenient to state equations (9) in terms of d; whereupon, setting

$$f_1'+\Delta+f_2=d,$$

we have $$p'=\frac{f_2'd}{d-(f_1'+f_2)} \quad \text{and} \quad p=\frac{f_1d}{d-(f_1'+f_2)}. \tag{9'}$$

p' is positive when P' lies to the right of P_2'.

p is positive when P lies to the left of P_1.

The *power* of an optical system. It is the custom in some fields of applied optics to call the reciprocal of the focal length the *power* of the system. The unit of power exists in the system possessing a focal length of one meter and is termed a *dioptre.* This method of designating focal lengths is particularly useful in numerical calculations, and the most common applications are found in connection with "thin" lenses (see page 26), especially in the computation of spectacle lenses.

In stating equations (8) in terms of powers, it will be convenient to substitute for Δ its value $d-(f_1'+f_2)$; whence, after transformation,

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2} \quad \text{or} \quad D=D_1+D_2-dD_1D_2, \tag{10}$$

where $D=\frac{1}{f}$, etc. A similar expression holds for D'.

There is little to be gained by a similar restatement of (9′) in terms of the reciprocals of p and d except in the case where $f_1=f_2$; whence, if $\frac{1}{p}=\phi$, the equation takes the form

$$\phi=D_1-2\left(\frac{1}{d}\right). \tag{11}$$

Combinations of thin lenses. If a system is formed of two thin lenses in air, we may apply equation (10), as follows (in this case $f_1=f_1'$, $f_2=f_2'$; see page 24):

1. *If* f_1 *and* f_2 *are positive* (and hence both lenses are convergent), the focal length of the combination has its smallest positive value when $d = 0$; that is, when the lenses are in contact. With increasing d, the focal length increases until, when $d = f_1 + f_2$, the focal length is infinite. This produces a *telescopic system.* For slightly greater values of d, f has large negative values, approaching zero with increasing d.

The values of f, as outlined above, mean little, however, unless the corresponding positions of the principal planes are known. An analysis of equation (9′) is therefore necessary in connection with that of (10) if the actual location of the principal foci is desired, and not merely the values of the focal lengths. Fig. 8 shows the positions of the principal foci and principal planes for different separations of a pair of identical equibiconvex lenses of focal length 2. Both the principal foci and the principal planes are at infinity for $d = 4$.

2. *If* f_1 *and* f_2 *are both negative* (and hence both lenses are divergent), the focal length of the combination has its largest absolute value (negative in sign) when $d = 0$. With increasing d, f approaches zero, remaining always negative.

3. *If one lens is convergent,* the other divergent (that is, $f_1 > 0$ and $f_2 < 0$), then for $d = 0$ the focal length of the combination is positive or negative according as f_1 is smaller or larger in absolute magnitude than f_2.

a. If $|f_1| > |f_2|$, f has its smallest negative value for $d = 0$ and grows continually larger in absolute value with increasing d, becoming infinite when $d = |f_1| - |f_2|$. Further increase in d causes decreasing positive values of f, approaching zero as d increases. See Fig. 9 for illustration of the case

$$f_1 = f_1' = 6,$$
$$f_2 = f_2' = -3.$$

In this case the principal planes and the principal foci are at infinity for $d = 3$. A familiar example of this type of telescopic system is found in the opera glass.

b. If $|f_1|<|f_2|$, f has its greatest positive value for $d=0$, and decreases with increasing d, remaining positive.

Of the above, two special cases, case 1 and case 3, *a*, are illustrated in Figs. 8 and 9 respectively. That lens which is shown in shaded outline is considered to be stationary, while the separations are effected by assigning various positions to the other, indicated by the unshaded outlines. The principal planes of the combination are indicated by vertical lines directed up from the axis, and the principal foci by lines directed down, each numbered to correspond to a certain separation of the lenses.

For example, in either figure, when the two components have the relative position indicated by the shaded figure and by the unshaded figure numbered 2, the emergent principal plane of the combination (considering the light to be incident from the left) is represented by the upper vertical line numbered 2, and the corresponding principal focus by the lower vertical line numbered 2. The principal focal length is, of course, the horizontal distance from the emergent principal plane to the principal focus.

Thin lenses in contact. This special case of the foregoing section deserves further notice. In case of contact, $d=0$, and equation (10) becomes $D=D_1+D_2$ (7). Had the system been composed of more than two components, the corresponding equation would have been $D=D_1+D_2+D_3+\cdots$ (8), or, in words, *the power of a combination of thin lenses in contact is the algebraic sum of the powers of the separate lenses.* With appropriate conventions as to sign (convergent lenses positive, divergent negative) this holds true for any combination of lenses under the restrictions designated by the title of this section.

This rule is in particularly common use for spectacle lenses. If a certain combination of spectacle lenses in contact (or nearly so) is found to produce distinct vision in a defective eye, they are replaced by a single lens, ground by the manufacturer to such specifications that its power is the sum of the powers of the separate lenses.

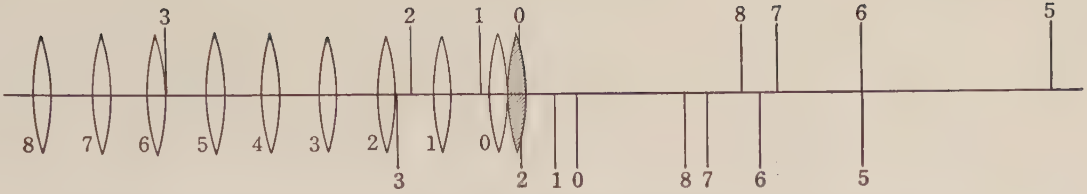

FIG. 8. Principal and focal planes of combinations formed by two converging lenses

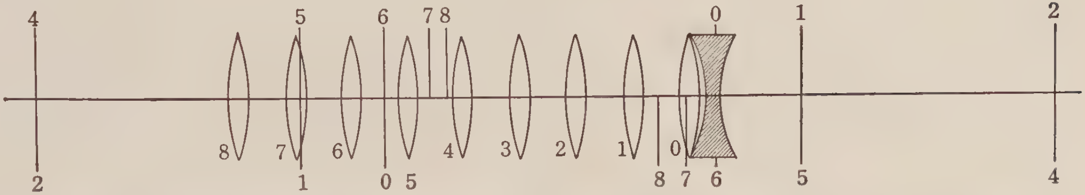

FIG. 9. Principal and focal planes of combinations formed by a converging and a diverging lens

EXPERIMENT I

CARDINAL POINTS OF A TWO-COMPONENT LENS SYSTEM

Apparatus. The equipment needed for this experiment has three main elements: (1) the lens system to be tested, (2) a wide beam of parallel light, and (3) a *nodal slide* to facilitate the location of the nodal points in the way described on pages 9 and 10.

1. The pair of lenses to be tested is so mounted as to allow ready variation and easy measurement of their separation. This is accomplished by mounting each in one end of a section of identical telescope tubing about one and one-half times its focal length. The positions of the principal planes are indicated by marks on the outside of the mounting.

These two sections of tubing are mounted in another section whose length is slightly less than the sum of the lengths of the others and whose diameter is sufficiently greater than that of the others to provide an easy sliding fit. By appropriate manipulation this combination will permit separations of the lenses from zero to more than twice the sum of their focal lengths. A slot cut lengthwise of the larger tube facilitates measurement of separations. The two lenses so mounted constitute an optical and mechanical unit, conveniently handled in the experiment to be described.

2. The wide beam of parallel light is best secured by a point source at the principal focus of a large, simple, converging lens. The source may consist of a gas-filled incandescent. The lens should have a diameter of at least three times that of the system to be tested.

3. The nodal slide is of the type described by Clay in his "Treatise on Practical Light." The rotating table should be not less than one meter long, and may be conveniently composed of two parallel meter sticks at such a distance apart as to allow the mounted lenses to slide along them. Fig. 10 shows the nodal slide and lens system as described.

A short-focus telescope equipped with a cross hair will be found convenient.

Adjustments. To test the parallelism of the beam of light, observe the source through the collimating lens with a telescope which has been previously focused for parallel light. Hold a densely colored glass before the eye to avoid possible damage to the eye resulting from the intensity of the image thus observed. Adjust the distance between the source and the lens until the image of the source as seen in the telescope is in good focus. As a final test, observe whether the apparent principal focal length of the lens to be tested remains the same for widely different distances from the collimating lens.

Measurements. 1. *Focal length of a converging lens.* Mount one of the lenses to be tested, on the nodal slide in the beam of parallel light, with its axis parallel to this beam. Catch the image on a screen, and measure the distance from this image (when it is in good focus) to the emergent principal plane of the lens. Take the mean of several such measurements as the focal length. Repeat with the other lens.

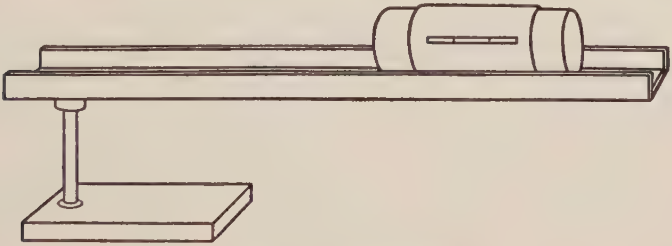

FIG. 10. Nodal slide

2. *Focal lengths of combinations of two biconvex lenses.* Focal lengths are to be measured from the appropriate principal points. The complete identification of the focal points of an optical system possessing appreciable thickness therefore requires two sets of measurements.

a. Determine the location of the emergent focal plane after the manner of the preceding section.

b. Determine the position of the emergent nodal point (and hence of the emergent principal point) in the way described on pages 9 and 10. Measurement of the distance between the two points thus determined will give the focal length. In this experiment the focal length should be thus determined for several distances between components.

In carrying out these measurements the following order of operations is advised:

1. Calculate the values of p' from equations (9′) or equation (11), for a fairly large number of values of d, from zero to the maximum

securable on the nodal slide, using the values of the focal lengths of the component lenses from the measurements described in section 1.

2. Calculate the focal lengths of the combination from equations (8) or (10) for the same values of d. Plot to scale the calculated positions of principal planes and foci for the different separations of the components, after the manner of Figs. 8 and 9.

3. Using the same apparatus as in section 1, but with the two lenses upon which calculations have been made, in place of one as before, separated by the successive values of d previously decided upon, locate positions of the principal foci of the combination.

4. Utilizing the rotating feature of the nodal slide, locate the emergent nodal points of the combination for the same values of d.

5. The corresponding principal focal lengths of the combination may now be measured.

6. The measured constants should now be charted in the same manner as the calculated constants, and measurements should be checked against calculations.

CHAPTER II

THE LENS

I. Lens Laws

Usual type of optical system. In the preceding chapter there were deduced some of the consequences of collinearity between object space and image space. It was noted that in deducing these consequences no assumptions were necessary concerning the physical characteristics of the systems involved. Some general conclusions were drawn concerning the characteristics of any system in which a collinear relation exists between object and image space; but to make these conclusions applicable to a given optical system, we must be more specific concerning the nature of the system involved.

Strictly speaking, the shapes of optical surfaces could be of wide variety. In practice, the use of spherical reflecting and refracting surfaces in the formation of images is universal. It is found that, with certain limitations, appropriate combinations of these surfaces may be made to meet the requirements of any optical instrument. We shall confine our attention to the effect of spherical refracting surfaces.

Refraction at a single spherical surface. It requires only a hasty sketch of two rays passing through a refracting spherical surface (see Fig. 14) to indicate the fact that not all rays from the same object point are brought to a focus at the same image point. But if we impose certain limitations on the angular aperture and on the divergence of the extreme incident rays, we may render this aberration as small as we please. We shall assume a spherical surface and an object which fulfill these conditions.

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Let OA (Fig. 11) be an incident ray and AO' the conjugate refracted ray. Let c be the center of the spherical surface. Then, applying the familiar "law of sines" of elementary trigonometry to the triangle OAC,

$$\frac{\sin\phi}{\sin\alpha} = \frac{PO + r}{AO},$$

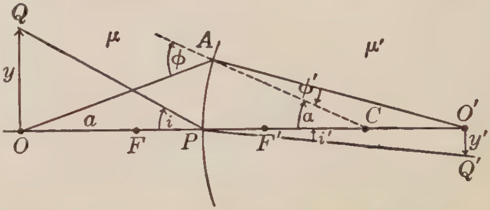


FIG. 11. Refraction at a spherical surface

where r represents radius of curvature; and similarly, in triangle $O'Ac$,

$$\frac{\sin\phi'}{\sin\alpha} = \frac{PO' - r}{AO'}.$$

Now, dividing,

$$\frac{\sin\phi}{\sin\phi'} = \frac{PO + r}{PO' - r} \cdot \frac{AO'}{AO}\left(= \frac{\mu'}{\mu}, \text{ by the law of refraction}\right). \quad (1)$$

Let us suppose that $AO = PO$ and $AO' = PO'$ to a sufficient degree of approximation. This leads to what is commonly termed the "first-order theory" of image formation by lenses. Call $PO = a$ and $PO' = a'$. Restating equation (1),

$$\frac{a + r}{a' - r} \cdot \frac{a'}{a} = \frac{\mu'}{\mu};$$

whence, by simple algebra,

$$\frac{\mu}{a} + \frac{\mu'}{a'} = \frac{\mu' - \mu}{r}. \quad (1')$$

Ray-tracing through optical systems. Most optical instruments may be regarded as a succession of spherical surfaces separating media of different index of refraction. Through such a system the course of light may be traced by successive applications of equation (1′). This equation adapts itself more readily to numerical computation when placed in a reciprocal form analogous to the second of equations (10) of Chapter I. Distances are measured from P, positive to the left in object space and positive to the right in image space. Radii of curvature are considered positive when the corresponding spherical surfaces are convex toward the left. These conventions assume that the incident light is traveling from left to right.

It should, perhaps, be remarked that the use of equation (1′) as a basis for ray-tracing is subject to the limitations of the first-order theory. The equation gives only approximate results in the sense that it cannot be used to detect aberrations.

Lateral magnification. Let $O'Q'$ (Fig. 11) be the image of OQ. It is readily shown that, under the conditions assumed, this image will be inverted. The lateral magnification has the value

$\frac{y'}{y} = \frac{a' \tan i'}{a \tan i}$. (For this case y and y' are of opposite sign.)

But it follows from the approximations $AO = PO$ and $AO' = PO'$ that the sines and tangents of the angles involved may be considered equal. Hence

$$\frac{y'}{y} = \frac{a' \tan i'}{a \tan i} = \frac{a' \sin i'}{a \sin i} = -\frac{\mu a'}{\mu' a}. \tag{2}$$

Therefore *the lateral magnification is proportional to the relative distances of image and object from the refracting surface*, the index of refraction of the initial and final media being the same, — a relation which may be shown to hold true with a slight modification for any optical system (see the second of equations (5′), Chapter I).

Principal planes. Equation (2) is identical in form with the second of equations (5′), on page 7; for, by equation (3′) of page 24, $\frac{f}{f'} = \frac{\mu}{\mu'}$. But in the case of equation (5′) distances were referred to the principal planes, and hence for a single refracting surface the *principal planes coincide and are tangent to the refracting surface* at the vertex.

Nodal points. A consideration of Fig. 11 shows that if $\mu \neq \mu'$, the only rays which pass without deviation through the refracting surface are those which are incident normally, and which therefore intersect the axis at the center of curvature. Hence for this case *the nodal points coincide at the center of curvature* of the refracting surface.

Principal focal lengths. Setting $a = \infty$ and $a' = \infty$ in turn in equation (1′), the solution of the resulting equations will give the following expressions for the principal focal lengths:

$$a_0' = f' = \frac{\mu' r}{\mu' - \mu}, \qquad a_0 = f = \frac{\mu r}{\mu' - \mu}. \tag{3}$$

Dividing one of equations (3) by the other gives the relation

$$\frac{f}{f'} = \frac{\dfrac{\mu r}{(\mu' - \mu)}}{\dfrac{\mu' r}{(\mu' - \mu)}} = \frac{\mu}{\mu'}, \tag{3'}$$

and hence *the ratio of the principal focal lengths is equal to the ratio of the indices of refraction* of the initial and final media. This law may be shown to be true for any system of coaxial refracting surfaces. Hence for the usual case (an optical system in air, where the refractive indices of the initial and final media are the same) the two focal lengths are equal, and, by page 11, *the principal points and the nodal points coincide.*

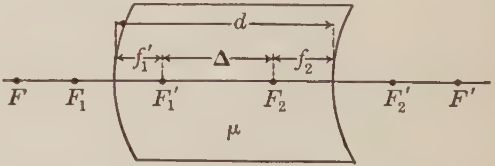


FIG. 12. Lens as a combination of two spherical surfaces

Lenses. We may derive some of the important laws of the simple lens by regarding the lens as a combination of two systems, each composed of a spherical refracting surface; in other words, by combining the processes of the foregoing section and of the last sections of Chapter I.

The radius of any surface will be considered positive when the surface is convex toward the incident light. We shall consider only the case of a lens of index μ in air of index unity.

In Fig. 12 the thickness (d) of the lens is manifestly

$$d = f'_1 + \Delta + f_2,$$

and, from (3),

$$f_1 = \frac{r_1}{\mu - 1}, \quad f'_1 = \frac{\mu r_1}{\mu - 1}, \quad f_2 = \frac{\mu r_2}{1 - \mu}, \quad f'_2 = \frac{r_2}{1 - \mu}. \qquad (4)$$

Substituting from these equations the values of f_1, f_2, and Δ in the second of equations (8) of Chapter I, we have, after reduction,

$$f = \frac{\mu}{\mu - 1} \cdot \frac{r_1 r_2}{d(\mu - 1) - \mu(r_1 - r_2)}. \qquad (5)$$

The value of f' as deduced from the first of the same pair of equations will be found to be the same as that of f, — as indeed it should be, according to the law deduced on page 24.

By substituting equations (4) into equations (9′) of Chapter I, it follows that

$$p = \frac{r_1 d}{(\mu - 1)d - \mu(r_1 - r_2)}, \quad p' = -\frac{r_2 d}{(\mu - 1)d - \mu(r_1 - r_2)}. \qquad (6)$$

Since the principal planes of a spherical refracting surface coincide with the plane tangent to the surface at the vertex (point where the surface is cut by the axis), p and p' must here represent the distances from the incident and emergent vertices of a lens to the incident and emergent principal points respectively.

Equations (6) therefore make it possible to calculate the positions of the principal planes of a lens whose thickness and radii of curvature are known, and with the aid of equations (5) to calculate also the positions of the principal foci. It is necessary when applying equations (6) to a given lens to keep in mind the convention concerning the sign of p and p'. As represented in diagrams in this text, it is that p is positive to the left, p' to the right.

A summary of expressions for focal lengths and positions of principal planes for standard lenses will be found on the next page. The refractive index is assumed to be 1.5. The approximate positions of the principal planes in certain representative lenses are shown in Fig. 13.

Thin lenses. The condition for thin lenses in equations (5) is that d be so small that $d(\mu-1)$ is negligible in comparison with $\mu(r_1-r_2)$. In this case

$$f=\frac{-r_1r_2}{(\mu-1)(r_1-r_2)} \quad \text{or} \quad \frac{1}{f}=(\mu-1)\left[\frac{1}{r_1}-\frac{1}{r_2}\right], \qquad (7)$$

which is the familiar *lens formula* of elementary optics.

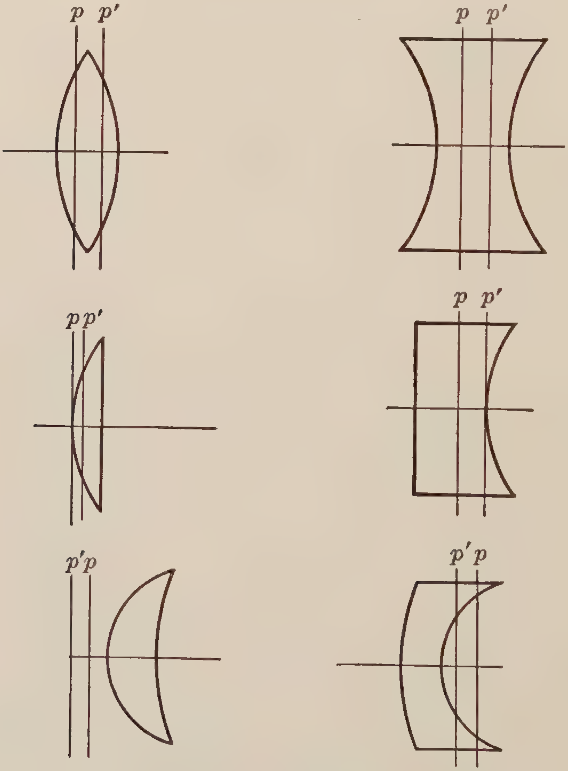


FIG. 13. Approximate positions of principal planes in standard lenses

Similarly, for this case, equations (6) become

$$p=-\frac{r_1d}{\mu(r_1-r_2)} \quad \text{and} \quad p'=+\frac{r_2d}{\mu(r_1-r_2)}. \qquad (8)$$

APPROXIMATE CHARACTERISTICS OF STANDARD LENSES

(Adapted from Gleichen)

KIND OF LENS	PRINCIPAL FOCAL LENGTH	DISTANCE OF PRINCIPAL POINTS FROM VERTICES OF SURFACE	REMARKS $\mu = 1.5$ (Use absolute values of r_1 and r_2 throughout)
Biconvex	$\frac{6r_1r_2}{3(r_1+r_2)-d}$	$p = -\frac{2r_1d}{3(r_1+r_2)-d}$ $p' = -\frac{2r_2d}{3(r_1+r_2)-d}$	Convergent, telescopic, or divergent. Principal points lie anywhere on optical axis.
Equibiconvex	$\frac{6r^2}{6r-d}$	$p = p' = -\frac{2rd}{6r-d}$	Convergent, telescopic, or divergent.
Positive meniscus	$\frac{6r_1r_2}{3(r_2-r_1)+d}$	$p = \frac{2r_1d}{3(r_2-r_1)+d}$ $p' = -\frac{2r_2d}{3(r_2-r_1)+d}$	Convergent always.
Plano-convex	$2r$	$p = 0$ $p' = -\frac{2d}{3}$	Convergent always.
Biconcave	$-\frac{6r_1r_2}{3(r_1+r_2)+d}$	$p = -\frac{2r_1d}{3(r_1+r_2)+d}$ $p' = -\frac{2r_2d}{3(r_1+r_2)+d}$	Divergent always. Principal points lie always in interior of lens.
Equibiconcave	$-\frac{6r_2}{6r+d}$	$p = p' = -\frac{2rd}{6r+d}$	Divergent always.
Negative meniscus	$\frac{6r_1r_2}{d-3(r_1-r_2)}$	$p = \frac{2r_1d}{d-3(r_1-r_2)}$ $p' = -\frac{2r_2d}{d-3(r_1-r_2)}$	Divergent, telescopic, or convergent.
Plano-concave	$-2r$	$p = -\frac{2d}{3}$ $p' = 0$	Divergent always.

II. Aberrations

Meaning of the term *aberration*. It was assumed early in this chapter that in the optical systems under consideration the aperture of the system and the divergence of the incident rays were sufficiently small to cause the formation of a distinct image. For most optical instruments, if simple lenses were used and stopped down to the point which distinctness of image requires, the amount of light available would be totally insufficient. Increasing the aperture, while it increases the brightness of the image, decreases distinctness, in addition to introducing several types of deformation of the image. This indistinctness and the accompanying deformations are commonly called *aberrations*.

Types of aberration. The primary assumption concerning the optical systems under discussion has been that of their ability to reproduce an object, point for point, line for line, and plane for plane. Needless to say, no optical instrument ever existed which could meet so rigorous a requirement.

It is the failure of an optical system to meet the requirement of point-for-point correspondence which causes indistinctness in the image. For points on or near the axis this indistinctness is commonly termed *spherical aberration*. For points at a considerable distance from the axis the image suffers further from the effects of *astigmatism* and *coma*. Failure to meet the line-for-line requirement gives rise to *distortion*; failure to meet the plane-for-plane requirement causes *curvature*.

In addition to the five aberrations already mentioned there exists *chromatic aberration*. This is a result of the difference in index of refraction for different wave lengths of light.

The way in which each of the foregoing aberrations manifests itself in the image is as follows:

Spherical aberration. The term *spherical aberration* is used in a variety of senses. We shall adopt it to denote the spreading of the image of a point source on the axis into a blur extending for a material distance along the axis in the image space.

This aberration is a consequence of the fact that the position of the image formed by reflection or refraction at a spherical surface depends on the *zone* within which the reflection or refraction

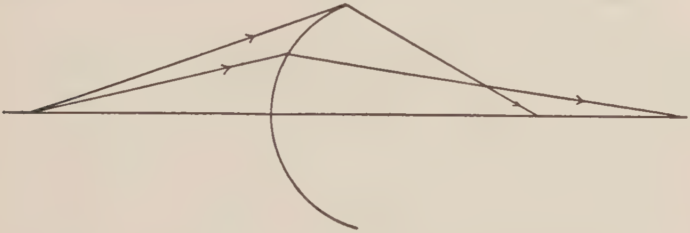


FIG. 14. Axial spherical aberration

occurs. The consequent spreading is illustrated, for the simple case of refraction at a single spherical surface, in Fig. 14, which illustrates a so-called *undercorrected* system. A lens interchanging the two refracted rays produces an *overcorrected* system.

Astigmatism. The phenomenon of astigmatism is illustrated in Fig. 15. The image of a point source not on the axis is in focus at two places, and at these two it is imaged not as a point but as a line. These two lines are perpendicular to the line

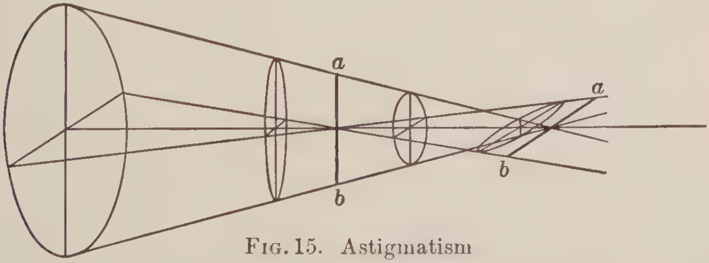


FIG. 15. Astigmatism

from object to image and to each other, and have the positions indicated in the figure by the two lines *ab*. The practical effect is that, in the case of the image of any object which presents the aspect of a right-angled screen, for the appropriate orientation of this screen the image of one of its sets of lines lies in one plane, while that of its other set lies in another.

Astigmatism is produced also (even for points on the axis) by lenses having different radii of curvature in different radial planes.

Coma. Coma usually manifests itself by the drawing-out of the image of a point source into an egg-shaped form. This egg-shaped image is the result of the superposition of a number of different-sized circular images not concentric (see Fig. 16). Coma may be attributed roughly to the fact that an object lying off the axis of a lens is not equidistant from all parts of any given zone of the lens surface.

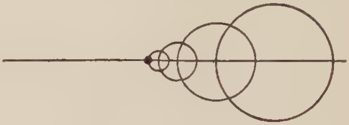
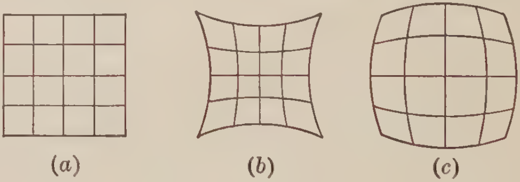


Fig. 16. Coma

Distortion. Distortion is a lack of similarity between object and image. For example, the inch marks on the distorted image of a foot rule or yardstick would not be equidistant from each other. Extending the same concept to the image of a right-angled net, there are two types of distortion, *barrel* and *pincushion.* In Fig. 17, (*a*) represents the object; (*b*), pincushion distortion of the image, which may be observed in a *virtual* image formed by a simple converging lens; and (*c*), barrel distortion of the image, which may be observed in the *real* image formed by a simple converging lens.



(*a*) (*b*) (*c*)

Fig. 17. Distortion

Curvature. The simplest case of curvature is the deformation of the image of a plane perpendicular to the axis into a curved surface symmetrical about the axis.

Chromatic aberration. The effect of chromatic aberration is indicated in Fig. 18. The image *a* of a point object in violet light is measurably nearer a converging lens than the image *b* of the same object illuminated by red light. When white light is used, an infinite number of such images are formed all the way from *b* to *a*. Consequently the image as observed on a screen placed anywhere between *b* and *a* will be blurred. The best position of the screen will be found to be at some point between *b* and *a*; but even at this location the image will not be a point, but a disc fringed by spectral colors.

Correction of aberrations. The design of lenses for use in telescopes, microscopes, cameras, and, in fact, all optical instruments consists largely in attempts to take advantage of the properties of different kinds of optical glass, as well as different shapes and relative positions of lenses, for the purpose of correcting the aberrations described above. The theory of these corrections would carry us far beyond the scope of this text, and cannot even be outlined here. It constitutes the major portion of most treatises on geometrical optics.

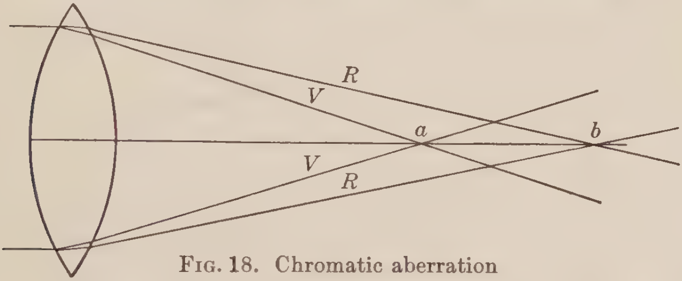


FIG. 18. Chromatic aberration

EXPERIMENT II

A. INDEX OF REFRACTION OF A CONVERGING LENS

Apparatus. An optical bench two meters long, with the following accessories, properly mounted for use with the bench:

Plano-convex lens, 15–25 cm. in focal length. The positions of the principal planes should be indicated by rulings on the mounting.

Ground-glass or white-cardboard screen.

Inclosed source of illumination. This will consist preferably of an electric lamp in a metal case. The front of this case should be plane, enameled white and provided with an aperture of about 2 cm. diameter with cross wires.

Plane mirror, capable of rotation about two axes in its own plane, at right angles with one another.

In addition to these optical-bench accessories there will be needed a spherometer, small steel scale, steel tape, and a pair of large "inside" calipers.

Adjustments. Arrange the source, lens, and mirror as in Fig. 19, with the plane side of the lens facing the source. The mirror should be as close to the lens as convenient. Its exact location is not important. The plane of the cross wires should be at a distance from the lens approximately equal to the principal focal length of the lens. (This may be determined with sufficient accuracy for preliminary purposes by causing the lens to form an image of some distant object, such as a spire or chimney against the horizon, and estimating or measuring roughly the distance from the lens to the image.) By appropriate manipulation of this distance and of the orientation of the mirror an image of the cross wires may be caused to fall upon the white surface of the lamp housing just to one side of the cross wires themselves. This image should be brought into exact focus.

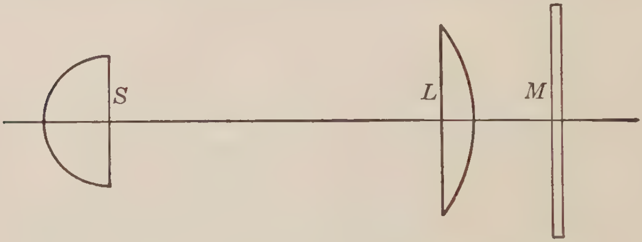


FIG. 19. Focal length by auto-collimation method

Measurements. *Measurement of principal focal length.* The distance from the plane of the cross wires to the nearest principal plane of the lens should now be measured with calipers for at least five independent settings. The mean of these measurements should give the principal focal length with considerable precision. It should not be necessary to explain why the distance so measured represents the principal focal length. The explanation is left as an exercise for the student.

Location of conjugate foci. Remove the mirror. Place the lens five or six times as far from the source as before, with its *convex* side toward the source.* Find the position of the image, and measure the distance from both image and object to the appropriate principal planes with the same degree of care as is prescribed in the preceding section. If a ground-glass screen is used, the plane of the image is on the *ground* side, irrespective of the way in which the screen faces.

Interchange object and screen, refocus, and repeat measurements. The mean of the measurements thus made should give quite accurately the values a and a' of two conjugate focal lengths. Apply the first of equations $(5')$ of Chapter I (noting that for this case $f = f'$), to find the value of the principal focal length. Compare this value with the one given by the previous method.

Index of refraction. Determine the radius of curvature of the lens. (If not familiar with the spherometer, refer to page 213.) Use the plane side of the lens as the surface upon which zero readings are to be taken. Substitute into equation (7) the measured values of r_1 and f, noting that r_2 is infinite. Calculate the value of μ. For a crown-glass lens this will come somewhere between 1.5 and 1.6.

B. THE ABERRATIONS OF A SIMPLE LENS

Apparatus. A testing-bench of the type represented in Fig. 20. L is the lens to be tested. It is mounted in such a way that it may be moved along the scale S_2, which should have a length greater than four times the focal length of the lens. On the same mounting as the lens is a diaphragm, with apertures as illustrated in Fig. 21. By turning the diaphragm about its axis, any one of the four apertures may be placed in front of the lens.

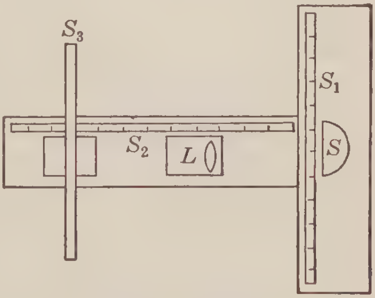


Fig. 20. Optical bench for observing aberrations

The source consists of a pair of illuminated cross wires. It should be mounted at the same height as the lens, and be so arranged that it may be moved along the scale S_1, which should have approximately the same length as S_2.

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* The general principle is that for a lens of this type the spherical aberration is a minimum when the plane face is on the side of the nearest conjugate focus. For the reason for this the student is referred to the following chapter.

It will probably be found of advantage to have a more intense source for some parts of this experiment than is furnished by the illuminated cross wires. The luminous filament of an incandescent electric light serves the purpose admirably for most of the observations, particularly if the lamp is so designed that its filament is nearly all in one plane.

The scale S_3 acts not only as a scale, but also as a screen to receive the image. It should be on a white background. To aid in focusing, the scale should be movable along S_2.

Measurements. Observation will be made of the aberrations described on pages 28–31, with the exception of coma, which cannot be conveniently observed without an approximate point source of considerable intensity.

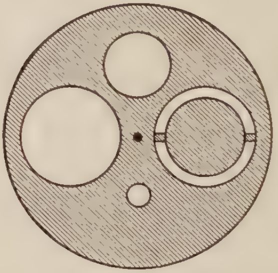

FIG. 21. Diaphragm for showing spherical aberration

Spherical aberration. Set the axis of the lens parallel to S_2, the lens being approximately at the center of S_2. Place the source approximately on this axis. Bring the smallest stop of the diaphragm opposite the center of the lens. Find by repeated trials the mean position of S_3, as measured on S_2, for which the image is in focus.

Cut away the central portion of the light with the appropriate stop, allowing only a ring of light around the margin of the lens to act, and repeat the above observation. The distance u between the two foci is termed the *longitudinal aberration* of the lens for this object distance, and for our purposes will be taken as the measure of the spherical aberration.

Astigmatism. Place the lens approximately at the center of S_2. Cause its axis to make an angle of roughly 30 degrees with S_2. Place the source on the optical axis of the bench. Manipulate S_3 until the image of the vertical wire of the source is in focus. Read the position of this point on S_2, taking the mean of several settings. Now find the position of S_3 for which the horizontal wire is in focus and repeat the readings. The distance v between the two positions of S_3 will be a measure of the astigmatism of the lens at that inclination.

Distortion and curvature. Set the apparatus as described under *Spherical aberration,* except that the full aperture of the lens is used. Note the position of the image of the vertical wire on S_3 and the position of S_3 on S_2. Move the source along S_1 five centimeters at a time, bringing the image back into focus for each new position, and note the corresponding settings on S_1 and S_2. Record the readings on all three scales for each setting, and represent the results graphically to scale as indicated in Fig. 22. The deviations of the intervals on the S_3 axis from equality give a measure of the distortion, and the magnitude of the S_2 distances a measure of the curvature of the plane of the image produced by the lens under the conditions of the experiment.

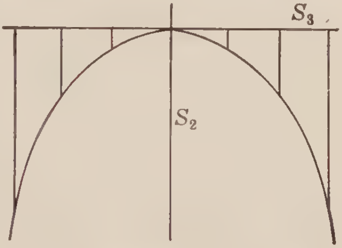

FIG. 22. Distortion and curvature

Chromatic aberrations. Set the apparatus as described under *Spherical aberration.* Turn the diaphragm to such a position that the light through the central portion of the lens is stopped out. Insert a red glass in the path of the light, preferably near the source. Bring the image of the cross wires carefully to a focus, by moving S_3, and note the position of S_3 on S_2. Substitute a blue glass for the red and repeat. The difference between the two positions of S_3 will be considered a measure of the chromatic aberration under the existing conditions.

CHAPTER III

THE SINGLE SLIT

Meaning of *diffraction.* The first two chapters deduced some of the consequences which follow from certain assumptions, among them that of the rectilinear propagation of light. It was noted in passing that this assumption was not justified by experiment, but that it was a sufficient approximation to fact to serve as part of the basis for a field of study known as geometrical optics. We shall now adopt the other point of view, that of physical optics, and shall first turn our attention to the consequences of departure from the assumption of rectilinear propagation.

The phenomena which occur in consequence of the deviation of light from the rectilinear path through other agencies than reflection and refraction are commonly known as *diffraction* phenomena. They usually occur in connection with the passage of light close to sharp edges of opaque bodies or through very narrow apertures. It is the latter class of phenomena that will mainly concern us.

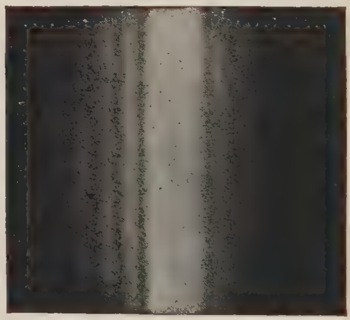

FIG. 23. Diffraction pattern produced by a single slit

Pattern due to a single slit. From the standpoint of *geometrical optics* the divisions between light and shadow on a screen placed behind a narrow slit that is illuminated from a slit source should be perfectly clear and distinct. The actual appearance of the field of view under these conditions is shown in Fig. 23.*

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* This figure is reproduced from "Light Waves and their Uses," by courtesy of Professor Michelson and The University of Chicago Press.

There is a slight difference in the appearance of the pattern thus produced, depending on whether the source and the screen are at a finite distance (spherical wave front; Fresnel class of phenomena) or at infinity (plane wave front; Fraunhofer class of phenomena). The latter case is the one which is most easily dealt with theoretically, and may be easily realized experimentally by placing both source and screen in the principal focal planes of converging lenses. It is the case which will now be considered.

Intensity function for single slit. Before undertaking the analysis presented in this section, the student will do well to refresh his memory regarding the customary mathematical methods of representation of a plane wave, as well as the laws of combination of simple harmonic waves (see Millikan and Mills' "Electricity, Sound, and Light," sects. 142 ff.).

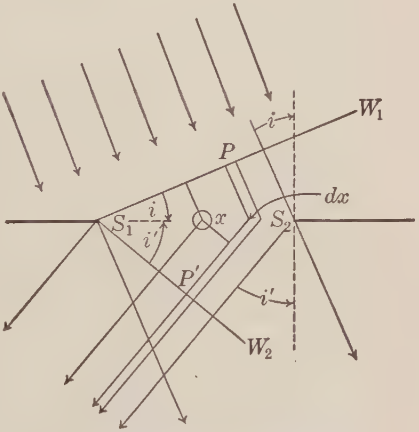

FIG. 24. Diffraction at a slit

Suppose (Fig. 24) a monochromatic plane wave W_1 to fall upon a screen at an angle of incidence i. This screen is pierced by a slit whose edges S_1 and S_2 are perpendicular to the plane of incidence, which is considered to be parallel to the plane of the paper. In consequence of Huygens' principle, illumination may be observed from the other side of the screen in directions other than that of the incident light. Suppose observation to be made from a direction making an angle i' with the normal to the screen and that W_2 is the diffracted plane wave observed. Let the width of the slit be a, and dx be an element of width of the slit at a distance x from the center, the center being the origin of coördinates. Assuming the vibrations which constitute the advancing wave front to be simple harmonic, they may be represented for a point P on the central ray by the customary expression $\sin \frac{2\pi t}{T}$, multiplied by an appropriate amplitude constant, and similarly, for some other point P' of the same wave front, by the expression

$$ds = cdx \sin 2\pi\left(\frac{t}{T} - \frac{\delta}{\lambda}\right). \tag{1}$$

cdx represents an amplitude factor, where c is a constant depending upon the intensity of the source, and where dx is the width of the slit element whose effect is being investigated. ds represents that part of the total displacement, at any point, that is due to the element of width dx of the slit. λ represents the wave-length of the incident light, and δ the difference between the lengths of path of the central ray and of the one which traverses the element of width dx in traveling from their positions in the W_1 plane to those in the W_2 plane. $\frac{\delta}{\lambda}$ therefore represents the phase difference between the vibrations in the corresponding two points of the W_2 plane. From the figure,

$$\delta = x(\sin i + \sin i').$$

For brevity let

$$\frac{\sin i + \sin i'}{\lambda} = \theta. \tag{2}$$

Then equation (1) takes the form

$$ds = cdx \sin 2\pi\left(\frac{t}{T} - \theta x\right). \tag{3}$$

We are considering the resultant light vibrations at the point where the image of the slit is being observed. The total displacement s, due to the whole slit-width, will be given by integrating (3) between the limits $-\frac{a}{2}$ and $+\frac{a}{2}$, with respect to the variable x. The result is

$$s = ca\frac{\sin \pi a\theta}{\pi a\theta} \sin 2\pi\frac{t}{T}. \tag{4}$$

The student is expected to verify this result.

This represents a simple harmonic vibration whose *amplitude* is

$$ca\frac{\sin \pi a\theta}{\pi a\theta}. \tag{5}$$

Intensity, being proportional to energy, is proportional to the square of the amplitude of the vibration causing the illumination. Hence the intensity at any point on the screen is given by

$$I=\left[ca\frac{\sin \pi a\theta}{\pi a\theta}\right]^2, \tag{6}$$

except for a proportionality factor which we may consider to have been "absorbed" into c. This is a function of θ, and therefore of i', the directions of the diffracted rays.

Equation (6) represents the intensity distribution which has already been illustrated in Fig. 23. This distribution is often represented graphically as in Fig. 25.

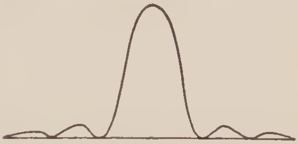

FIG. 25. Intensity pattern due to single slit

Minima and maxima of intensity. From equation (6) may be secured information concerning the location and values of the minima and maxima of the diffraction pattern represented in Figs. 23 and 25. The smallest value that the second member of the equation can assume is plainly zero; and zero is, in fact, the value of the intensities at all the minima. The values of θ for these points may be deduced by inspection of the equation. It will be seen that $I=0$ when $\pi a\theta = m\pi$ (where m has any integral value except zero); that is, when $\theta=\frac{m}{a}$, or, by equation (2),

$$\sin i'=\frac{m\lambda}{a}-\sin i. \tag{7}$$

Hence successive minima occur on one side of the central maximum when m assumes the values 1, 2, 3, $\cdots$, and on the other side for -1, -2, -3, $\cdots$. The distance between the two central minima corresponding to $m=1$ and $m=-1$ is twice that between any two other adjacent minima.

The positions of *maximum* illumination *do not* lie midway between these zero minima. Their location may be found by the usual method of equating to zero the first derivative of the function under investigation. Disregarding the constants, equation (6) may be expressed in the form

$$I = \frac{\sin^2 u}{u^2}, \tag{8}$$

where

$$u = \pi a \theta.$$

Whence

$$\frac{dI}{du} = 2\,u \cdot \frac{\sin u}{u} \cdot \frac{u \cos u - \sin u}{u^3} = 0. \tag{9}$$

This equation is satisfied if either

1. $u = 0$, which plainly gives the condition for the central maximum; or
2. $\frac{\sin u}{u} = 0$, which is readily seen to give the minima previously discussed; or
3. $\frac{u \cos u - \sin u}{u^3} = 0$, whence $u = \tan u$. (10)

The readiest way to solve equation (10) for u is to plot the equations $y = u$ and $y = \tan u$ and observe the points of intersection (Fig. 26). A method which gives greater accuracy was used by Schwerdt,* who found the values given in the second column of the table on the opposite page.

The relative intensity of each of these maxima appears in the third column, on such a scale that the central maximum has an intensity 1. The rapidity with which these intensities fall off in each direction from the center is evident, the central maximum having more than twenty times the intensity of the adjacent maxima.

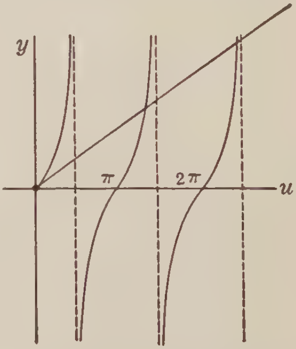

FIG. 26. Graphical solution of equation (10)

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* Schwerdt, Beugungserscheinungen, p. 145, Table No. 1. 1835.

In the fourth and fifth columns appear the relative positions of the maxima and minima on such a scale that the first minimum is unit distance from the central maximum. The fact that the maxima are not midway between the minima (except in the case of the central maximum and its adjacent minima) will be shown by an examination of these two columns. That the maxima approach more and more nearly to a position midway between their adjacent minima as we proceed outward from the central maximum will also be evident. The essential facts outlined by the table are portrayed graphically in Fig. 25.

n	u_n	I	Position of Maximum	Position of Minimum
0	0.000	1.0000	0.000	
				1
1	1.430π	.0469	1.430	
				2
2	2.459π	.0168	2.459	
				3
3	3.471π	.0083	3.471	
				4
4	4.477π	.0050	4.477	
				5
5	5.482π	.0034	5.482	
				6
6	6.484π	.0024	6.484	
				7
7	7.486π	.0018	7.486	

Resolving power of a single slit. The foregoing considerations indicate that the central maximum of an image whose size would otherwise be altogether inappreciable may be spread by diffraction to an appreciable size. If *two* sources exist, separated by a small angular interval, the central maxima of their images may or may not overlap. If they overlap to a sufficient extent, they become indistinguishable from each other and, as commonly expressed, are not resolved. The term *resolving power* in general signifies the ability of an optical instrument to separate two closely situated images.

The amount of overlapping which can exist without loss of resolution depends largely on the acuteness of vision of the observer. The commonly accepted criterion (due to Rayleigh) is, however, that the limit of resolution may be considered to have been reached when the central maximum due to one source falls on a first minimum due to the other (Fig. 27). The angular separation of the two maxima is then $\frac{\lambda}{a}$ (see equation (7)). The resultant intensity distribution is indicated by the dotted line. The student may show that the intensity at the mid-point of this line is $\frac{8}{\pi^2}$ times that of either maximum when the two maxima are equal.

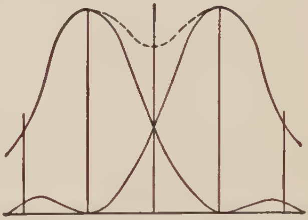

FIG. 27. Limit of resolution of single slit

EXPERIMENT III

WAVE-LENGTH BY SINGLE-SLIT DIFFRACTION PATTERN

Apparatus. This experiment is most easily performed with a reading-telescope, preferably one somewhat larger than the usual type. It should be provided with a slit of variable width, mounted in front of the objective in such a way that it may be removed and replaced at will. The telescope should be so mounted that it is capable of rotation about a vertical axis. Measurements will be greatly facilitated if there is provided a slow-motion screw for effecting this rotation. A mirror mounted at any convenient point on the telescope, together with an auxiliary reading-telescope and scale, constitutes the best means of measuring the angle through which the large telescope rotates.

The best source of illumination is the intense sodium light produced by placing the tip of a hard-glass rod in an oxy-gas flame. This should be converted into a slit source by mounting a coarsely adjustable slit in front of it. As a second choice, a single-filament electric light viewed through a red glass will be found tolerable. Whichever source is used, it should either be "placed at infinity," by a collimating-lens, or stationed a sufficient distance away, so that light from it may be considered essentially parallel. A distance of ten times the focal length of the telescope used is sufficient.

A comparator or calibrated micrometer microscope will be necessary, to measure the width of the slit (see page 214).

Adjustments. Arrange all apparatus at the same level. Remove the slit from the telescope and focus the telescope on the source. Replace the slit, adjust it to parallelism with the source, and narrow it to the point where diffraction fringes become visible. Make such adjustments of direction, width of slits, height of flame, etc. as are most effective in securing maximum visibility of the fringes. Adjust telescope and scale for measurement of the angle through which the large telescope turns. If not familiar with this method of angle measurement, consult page 216.

Measurements. A precise measurement of the angular width of one of the fringes and of the corresponding width of the slit is all that is required. Since the former is a very small quantity, it is better to measure the angular width of a counted large number of fringes, and hence to deduce the width of a single fringe. In all measurements settings should be made on the dark portion of the fringes. (Why?) In passing across the central fringe count it as *two*.

The following procedure is recommended in counting: Suppose there are twenty fringes on which readings can be made. Make careful settings of the cross hair on each of the first five, bringing it up at least five times to the center of each fringe, always from the same side. Repeat for the last five, carefully counting the intervening fringes, but not making settings on them. Then subtract the reading of the first from that of the sixteenth, similarly for the second and seventeenth, etc. Each difference will constitute an independent determination of the width of fifteen fringes, from the average of which a quite reliable value of the width of a single fringe may be deduced.

Measure a, the width of the slit.

Substitute this value of a and the value i' of the angular width of a single fringe in equation (7), in which $i = 0$, $m = 1$, and i' is small enough to justify the approximation $\sin i' = i'$, where i' is expressed in radians. With these substitutions equation (7) becomes

$$\lambda = i'a.$$

Repeat the measurements for a different value of a.

CHAPTER IV

THE DOUBLE SLIT

Pattern due to two slits. The substitution of two similar parallel slits for the single one discussed in the preceding chapter will have two effects. The first will be to intensify the pattern already existing.* The maxima due to one slit will be superposed on those due to the other, provided that the two slits have the same width, irrespective of the distance between them.

FIG. 28. Diffraction and interference produced by a double slit

The second effect will be the appearance, in the center of these diffraction fringes, of a second set of fringes, much finer and sharper, due to interference between the beams from the two slits. Fig. 28 † shows the appearance of the field under these circumstances. In this photograph the source, instead of being monochromatic, was white light. The appearance if photographed by monochromatic light would have differed, however, in only minor respects from that of the illustration.

Applications. These interference fringes possess a twofold significance. First, they form the beginning of the phenomenon which, with an increasing number of slits, develops into *grating*

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* The student should avoid confusion between the pattern due to a single slit from a double *source*, as discussed in the latter part of the preceding chapter, and that due to a double *slit* from a single source.

† This figure is reproduced from "Light Waves and their Uses," by courtesy of Professor Michelson and The University of Chicago Press.

spectra; and, second, they have become of increasing importance in the astronomical world through the use that has been made of them in the study of the orbital motions of double stars, the sizes of the minor planets, the diameters of the nearer stars, and similar problems, which lie almost hopelessly beyond the power of the best telescopes.

Intensity function for double slit. The procedure in the derivation of the intensity function for two slits of width a, separated by an opaque portion of width b, is identical with that for a single slit. The difference in result proceeds from the fact that the integral of equation (3) of Chapter III must be taken between the limits $\left(\frac{a}{2}+b\right)$ and $\left(\frac{a}{2}+b+a\right)$ as well as between $-\frac{a}{2}$ and $\frac{a}{2}$, and the results added. The final expression for the intensity is

$$I=\left[2\,ac\frac{\sin\pi a\theta}{\pi a\theta}\right]^2\cos^2[\pi(a+b)\theta]. \qquad (1)$$

The student is expected to verify this by performing the integration.*

The first factor on the right side of equation (1) is the same as equation (6) of Chapter III, except that the intensity is four times (the amplitude twice) that expressed in equation (6), owing to the fact that there are now two slits, each of width a. This is the first effect mentioned on the preceding page. The nature of the second effect, described by the second factor of equation (1), may be investigated as follows:

Maxima and minima of intensity. This second factor possesses zero *minima* whenever $\pi(a+b)\theta$ becomes equal to any

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* There will be needed in the reduction of these integrals the familiar relations from elementary trigonometry:

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2},$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

odd multiple of $\frac{\pi}{2}$, that is, whenever $\pi(a+b)\theta = (2m+1)\frac{\pi}{2}$; whence $\theta = \frac{2m+1}{2(a+b)}$, where $m = 0, 1, 2, 3, \cdots$; and hence (see equation (2), Chapter III) for values of i' such that

$$\sin i' = \frac{(2m+1)\lambda}{2(a+b)} - \sin i. \tag{2}$$

This factor possesses *maxima* whenever $\pi(a+b)\theta$ equals any integral multiple of π, that is, whenever $\pi(a+b)\theta = m\pi$; whence $\theta = \frac{m}{a+b}$, where $m = 0, 1, 2, 3, \cdots$; and hence for values of i' such that

$$\sin i' = \frac{m\lambda}{a+b} - \sin i. \tag{3}$$

Comparison of (2) above with (7) of Chapter III will show that the minima are closer together than those due to a single slit, the intervals in the two cases having the ratio $\frac{a}{a+b}$.

Similarly, comparison of (2) with (3) will show that, unlike the pattern from a single slit, individual fringes in this case are symmetrical, maxima lying midway between minima, except as the double-slit intensity is influenced by the single-slit distribution which is still present (first effect).

Discussion of the relative intensities of successive double-slit maxima will be simplified by considering two special cases: (1) that in which the distance $a+b$ between slit centers is maintained constant, and only the width a of the individual slits is altered; (2) that in which the width a of the individual slits is maintained constant, and only the distance $a+b$ between centers is altered.

In the first case the *positions* of the double-slit fringes will remain unchanged; for consideration of equations (2) and (3) shows that i' is not affected in either the maxima or minima by any change in a which leaves the value of $a+b$ unchanged. The *relative intensities* of the maxima *will* change, however, depending on the part of the single-slit pattern in which they lie as that pattern, spreading or contracting with changes in a, moves over them. Except for the intensity distribution imposed by the single-slit pattern, these fringes are all of the same intensity.

In the second case the positions of the double-slit fringes *will* change, in proportion to changes in $a + b$. The intensities of their maxima will also change, for these intensities can never exceed that of the portion of the single-slit pattern in which they lie.

It will be seen that equation (1) is such a function of a and $a + b$ that the single-slit pattern is the envelope of the double-slit pattern. Fig. 29, taken from Schwerdt, illustrates the intensity distribution in the diffraction pattern due to a double slit in which the opaque portion has twice the width of one of the slits.

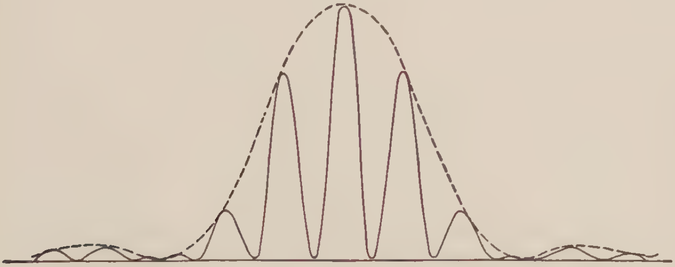

FIG. 29. Intensity distribution due to two slits

For light which is rigorously monochromatic the double-slit pattern should cover the entire field of the single-slit pattern. Practically, however, it is seldom discernible in more than a very few of the maxima of the latter, and, for white light, in only a portion of the central maximum.

If the width of the slits is sufficiently small (of the order of a few light waves), the single-slit pattern becomes spread out to such an extent that its central maximum is the only part discernible, and that hence the double-slit maxima continually fall off in intensity until they are lost. This is usually the case in the diffraction grating.

Missing orders. The successive maxima on each side of the central maximum of the double-slit fringes are called maxima (or spectra) of the first, second, etc. *orders.* Interesting special cases occur when the fraction $\frac{a+b}{a}$ has integral values. When it has the value 2 (that is, when the opaque distance between the slits is equal to the width of each slit), the orders of even number are absent. The student will show the reason for this, and deduce the consequences of this fraction's having the values 3 and 4.

Resolving power of a double slit. The angle subtended at the plane of the slit by successive maxima or minima may be seen from equations (2) or (3) to be $\frac{\lambda}{a+b}$. If, now, the double slit is illuminated by two sources separated by a small angular interval, the remarks of the last paragraph of Chapter III are again applicable. There is, however, the significant difference that when a maximum that is due to one source falls on an adjacent minimum that is due to the other, the angular separation of the two sources is $\frac{\lambda}{2(a+b)}$ in place of $\frac{\lambda}{a}$, as in the case of the single slit. There follows the very interesting consequence that the resolving power of a single rectangular opening of width d is only half that of *two* such openings having the *same* distance d between their centers. Hence the resolving power of a telescope may be *increased* by covering the central portion in the manner illustrated in Fig. 30 and by allowing light to enter only at the two uncovered portions. In this way additional resolving power may be secured from any telescope, at the expense, of course, of brightness of the image. The central section of the lens may be removed, or, in the extreme case, the entire lens may be replaced by four mirrors, arranged with a smaller auxiliary telescope, as in Fig. 31.

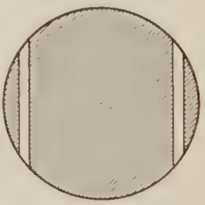
FIG. 30. Method of increasing resolving power of a telescope

Application to measurement of stellar magnitudes. The angle subtended at the earth by even the largest and nearest of the stars is so minute that it lies beyond the resolving power of our largest telescopes. This is also true of some of the smaller satellites of Jupiter and the orbits of some of the double stars. Important contributions have been made to our knowledge of stellar magnitudes by increasing the resolving powers of astronomical telescopes in the manner of Fig. 30. Measurements of this type were first made by Professor A. A. Michelson of The University of Chicago, at the Lick Observatory, in 1900.*

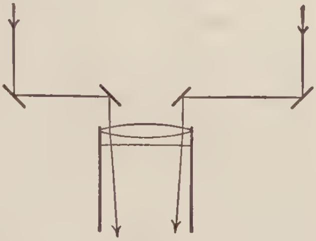

Fig. 31. Principle of Mt. Wilson stellar interferometer

The objective of a small telescope was covered, except for two slits whose distance apart could be varied. The instrument was then directed toward the object under observation (in this case one of Jupiter's satellites whose diameter was sought), and the distance between the slits was altered until the double-slit fringes disappeared. The angular diameter of the object was then given by $1.22\frac{\lambda}{d}$,† λ being the effective wave-length of the light, and d the separation of the two slits covering the objective. It was found possible, by this method, to make a more accurate determination of these magnitudes with a six-inch telescope than with a forty-inch telescope employed in the usual method of direct observation. A number of years later, similar methods gave excellent results in the determination of the orbits of close visual binaries, that is, double stars revolving around their common center of mass.

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* Michelson, Light Waves and their Uses, chap. vii.

† When the source is a circle of appreciable diameter, the condition for disappearance of the double-slit fringes may be shown to be given by this expression, in place of $\frac{\lambda}{d}$ which holds for two adjacent point or slit sources.

By the modification of this method indicated in Fig. 31, Professor Michelson made possible the first direct measurement of a stellar diameter, some twenty years later (1921).* The mirrors were mounted on a frame permitting a separation of twenty feet, and the frame was mounted on the hundred-inch Mount Wilson reflector. The fringes from α Orionis were found to disappear when the separation of the mirrors was 121 inches, giving a value of $0''.047$ for the angular diameter of this star. To make this measurement by the ordinary methods of astronomy with the same degree of precision would require a telescope with an objective of over one hundred feet in diameter, which is, of course, out of the question.

EXPERIMENT IV

THE ANGULAR DIAMETER OF A DISTANT SOURCE

Apparatus. The most important piece of apparatus for this experiment is a long-focus lens. A telescope objective of from 1 to 2 meters focal length and 5 or 6 inches in diameter is the best. Such lenses are expensive, however, and not vitally necessary. Quite acceptable results may be secured by an uncorrected converging lens with about the same characteristics. With such a lens it will be advisable to render the source approximately monochromatic by a color filter. Other aberrations will not be troublesome if attention is confined to the center of the field.

A high-power eyepiece should be used; or, better, in place of a simple eyepiece, a compound microscope.

The double slit may consist of two parallel identical slits cut in a piece of cardboard. In this case several such cardboards should be cut, with 1-mm. slits separated by 1, 2, 3, 4, and 5 cm. respectively. It is far preferable, however, to construct a double slit, so arranged that the distance between the two slits may be continuously varied from 1 to 5 cm. by means of a right-and-left screw or some other mechanism which moves them symmetrically with respect to the center of the lens.

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* Michelson and Pease, in *Astrophysical Journal*, Vol. 53, No. 249 (1922).

A very intense source is essential to the success of this experiment. A right-angled carbon arc will be found satisfactory. In front of this should be placed a slit of variable width.

Adjustments. Fig. 32 shows the arrangement of equipment. The distance from source to point of observation should be not less than ten times the focal length of the lens. The double slit should be placed just in front of the objective, and the microscope or eyepiece just back of its principal focus. It is advisable to mount lens, eyepiece, and double slit together in one unit, making essentially an astronomical telescope with the addition of a double slit. Any color filter that is used should be placed as near the source as practicable or may be held before the eye of the observer. It should *not* be placed near the objective. In finding fringes for the first time, remove the double slit, bring the source slit into good focus, and then replace the double slit, setting it at a separation somewhere near its maximum. Adjust the width of the source slit until the interference fringes appear. Care must be taken to distinguish between the coarse diffraction fringes and the very fine and sharp interference fringes, which will be the most distinct within the central bright fringe of the diffraction pattern. Constant adjustment of the arc will be found necessary, to keep the intensity sufficiently high for easy observation.

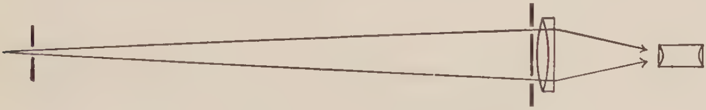

FIG. 32. Arrangement of double-slit experiment

Measurements. Two types of measurements will be taken, both having for their object a determination of the angular width of the source slit by the method described on page 49. One method involves varying the separation of the two portions of the double slit, maintaining the source slit at constant width; the other, varying the width of the source slit, maintaining the double slit at constant separation. The latter method will, of course, be the only one available if the double slit is not variable.

First method. Set the source slit at a width of approximately one-half millimeter. Decrease the distance between the two parts of the double slit until the interference fringes disappear. If this

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is a first-order disappearance, fringes, having reappeared as the two portions of the double slit are brought still closer together, will not again disappear. If disappearance does occur, continue the process until you have arrived at the separation corresponding to the first-order interference. When this point is reached, measure the distance between the centers of the two portions of the double slit, separate them, and repeat the operation. In this way take not less than five measurements, using the mean d of these measurements in subsequent calculations. If white light is used, the value 0.000055 cm. may be used for λ. If a color filter is used, the instructor will give the effective wave-length transmitted. Measure the distance l from the source slit to the double slit. The width a of the source slit is then given by the relation

$$a=\frac{l\lambda}{d}. \tag{4}$$

The width of the source slit should now be measured with a micrometer microscope as a check (see page 214). The allowable error will vary from 1 to 5 per cent, depending upon the conditions of the experiment. Repeat observations for another value of the width of the source slit.

Second method. Set the two portions of the slit at a fixed distance apart, near the maximum obtainable. Continuously vary the width of the source slit and note the successive appearance and disappearance of the fringes. Measure the width of this slit with a micrometer microscope at as many successive points of disappearance as can be observed. These widths will be found to be multiples of a certain value. The mean value of this greatest common divisor should agree with the value of a calculated from equation (4).

CHAPTER V

N SLITS

Pattern due to *N* slits. The patterns due to 3, 4, ⋯ equidistant and identical slits are readily derivable by further extensions of the limits of integration of equation (3) of Chapter III. Four of the corresponding intensity curves are represented in Fig. 33. Instead of dealing with these special cases, however, we shall proceed to the general case of the diffraction pattern due to *N* slits. We may then make applications to the cases mentioned, if we choose, by giving *N* the appropriate values in the resulting relations.

The diffraction grating. The case of *N* slits, where *N* represents a large number, is of particular interest on account of its application to what is known as the *diffraction grating.* The grating is one of the most powerful appliances in the field of spectroscopy. In consequence of some properties which will be discussed presently, even a small grating is likely to be more effective for many purposes than the best of prisms. In its usual form it consists of a well-polished metal surface upon which are ruled from twelve to fourteen thousand lines for each inch of width. The excellence of the grating is determined by its size, the closeness and uniformity of its ruling, and the shape of the grooves constituting it.

The performance of the grating will be indicated by a study of Fig. 33, *d.* The monochromatic slit source has been spread into a number of lines (successive orders of spectra) whose intensities diminish as we proceed either way from the center. Had the source been other than monochromatic, each line would have been replaced by the spectrum characteristic of the source, the successive spectra becoming broader as they became dimmer, until those of the higher orders would have been so overlapped as to constitute almost complete confusion.

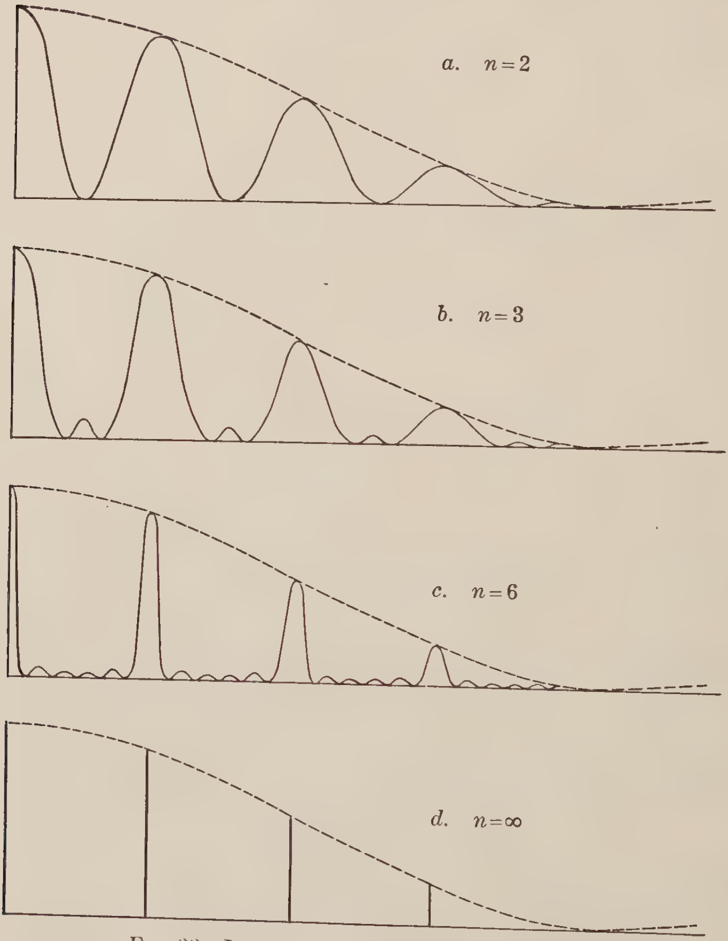

FIG. 33. Intensity distribution due to n slits

Intensity function for N slits. To find the values and locations of the maxima and minima for a large number of identical, equidistant, and closely spaced slits, we need only to carry out the logical extension of the process applied to one and two slits. That is, we shall integrate equation (3) of Chapter III between the successive limits

$$-\left(\frac{a}{2}\right) \text{ and } \left(\frac{a}{2}\right), \quad \left(\frac{a}{2}+b\right) \text{ and } \left(\frac{a}{2}+b+a\right),$$

$$\left(\frac{a}{2}+2\,b+a\right) \text{ and } \left(\frac{a}{2}+2\,b+2\,a\right), \quad \ldots,$$

and add. We shall have, upon completion of this process, the following trigonometric series:

$$\begin{aligned} ac\frac{\sin \pi a\theta}{\pi a\theta}\Bigg\{\sin 2\,\pi\frac{t}{T} + \sin 2\,\pi\left[\frac{t}{T}-(a+b)\,\theta\right] \\ + \sin 2\,\pi\left[\frac{t}{T}-2\,(a+b)\,\theta\right]\cdots \\ + \sin 2\,\pi\left[\frac{t}{T}-(N-1)\,(a+b)\,\theta\right]\Bigg\}. \end{aligned} \qquad (1)$$

The student may set up the limits corresponding to the *K*th slit, perform the integration, deduce the corresponding term of the series, and from this prove the correctness of expression (1).

The series part of this expression is of the form $\sum\limits_{0}^{n}\sin(\alpha - k\beta)$, and its sum is *

$$\frac{\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}\sin\left(\alpha-\frac{n-1}{2}\beta\right)$$

and therefore (1) becomes

$$ac\frac{\sin \pi a\theta}{\pi a\theta}\,\frac{\sin N\pi\,(a+b)\theta}{\sin \pi\,(a+b)\theta}\sin 2\pi\left[\frac{t}{T}-\frac{N-1}{2}(a+b)\theta\right], \quad (2)$$

and the intensity is therefore

$$I=(ac)^2\left[\frac{\sin \pi a\theta}{\pi a\theta}\right]^2\left[\frac{\sin N\pi\,(a+b)\theta}{\sin \pi\,(a+b)\theta}\right]^2. \qquad (3)$$

---

* Todhunter, Plane Trigonometry (edition of 1890), p. 241.

Minima and maxima of intensity. The second factor of equation (3) represents the intensity due to the single-slit pattern of each grating aperture. This we have already studied. Turning to the third factor, let

$$\left[\frac{\sin N\phi}{\sin\phi}\right]^2 = Y^2, \tag{4}$$

where

$$\phi = \pi(a+b)\theta. \tag{4'}$$

The condition for the existence of maxima and minima of intensity is $\frac{d(Y^2)}{d(\phi)} = 0$. Hence we write

$$\frac{d(Y^2)}{d\phi} = \frac{2\sin N\phi}{\sin\phi} \times \frac{N\sin\phi\cos N\phi - \sin N\phi\cos\phi}{\sin^2\phi} = 0. \tag{5}$$

This equation has three solutions.

(*a*) $$\sin\phi = 0.$$

Equations (4) and (5) are indeterminate for this case. The student may show that (*a*) is, nevertheless, a solution of equation (5) and that (4) becomes

$$Y^2 = N^2. \tag{6}$$

This solution gives a series of *principal maxima* whose intensities are much greater than those given by solution (*c*) below. When N is very large (as is usually the case for a grating), these maxima are the only ones visible. The corresponding values for θ, and hence for i', are given by

$$\phi = \pi(a+b)\theta = m\pi, \qquad m = 0, 1, 2, \cdots;$$

whence (see equation (2), Chapter III), substituting for the "grating space" $a+b$ the term d,

$$\frac{d(\sin i + \sin i')}{\lambda} = m$$

or

$$\left.\begin{aligned} d(\sin i + \sin i') &= m\lambda \\ \sin i' &= \frac{m\lambda}{d} - \sin i. \end{aligned}\right\} \tag{7}$$

or

The first of equations (7) states that for these maxima the path difference between rays from two consecutive slits is an integral number of wave-lengths of light.

(*b*) $\sin N\phi = 0$, where $\sin \phi \neq 0$.

This solution will be readily seen to indicate a series of zero minima, $(N-1)$ in number, between successive principal maxima. The corresponding values for θ, and hence for i', are given by

$$\phi = \pi(a+b)\theta = \frac{m\pi}{N}, \text{ where } m \neq 0;$$

and hence, letting $a + b = d$ as before,

$$\left.\begin{aligned} d(\sin i + \sin i') &= \frac{m\lambda}{N} \\ \text{or} \qquad \sin i' = \frac{m\lambda}{Nd} - \sin i &= \frac{m\lambda}{l} - \sin i, \end{aligned}\right\} \qquad (8)$$

where l represents the width of the grating.

(*c*) $N \tan \phi = \tan N\phi$.

This equation follows immediately by equating the numerator of the second factor of (5) to zero. It includes (*a*) as a special case. It may be solved for ϕ by a method similar to that described for the solution of equation 10 of Chapter III. It gives, besides the principal maxima, a series of subsidiary maxima, $(N-2)$ in number, between successive principal maxima. These subsidiary maxima are of unequal intensity and are not equidistant. When N is large, the ratios of their intensities to those of the principal maxima become vanishingly small. They are therefore not visible in the ordinary grating spectrum. For this reason it is not profitable to enter into any detailed consideration of their characteristics. Their general features appear in Fig. 33.

Minimum deviation. There is a certain angle of incidence for which the deviation produced by a grating possesses a minimum value. Reference to Fig. 24 will show that the deviation δ *

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* This δ is not to be confused with that of Chapter III, which signifies path difference.

produced by a grating is given by the relation $\delta = i + i'$. To find the value of i for which δ possesses a minimum value, restate equations (7) in the form

$$\frac{m\lambda}{d} = 2 \sin\frac{i+i'}{2} \cos\frac{i-i'}{2}, \tag{9}$$

whence

$$\delta = i + i' = 2 \sin^{-1} \frac{m\lambda}{2\, d \cos\frac{i-i'}{2}}. \tag{9'}$$

Inspection of equation (9′) will show that δ possesses a minimum value when $\cos\frac{i-i'}{2} = 1$; that is, when $i = i'$. The condition for minimum deviation in the grating is hence the same as in the case of the prism (compare equations (7), Chapter VI).

For this case, equation (9) takes the simplified form

$$\sin\frac{\delta}{2} = \frac{m\lambda}{2\,d}, \tag{9''}$$

where δ now represents the angle of *minimum deviation.*

The application of this to measurements with the grating is described in the experimental part of this chapter. The definition of spectral lines is the best when the grating is used in the position of minimum deviation.

Width of principal maxima. The distance Δ between the zero minima on each side of a principal maximum is given by the equation $\Delta = i_1' - i_2'$, where i_1' and i_2' are the values of i' corresponding to $m = 1$ and $m = -1$ in the last of equations (8). These substitutions give the two equations

$$\sin i_2' = -\frac{\lambda}{l} - \sin i$$

and

$$\sin i_1' = \frac{\lambda}{l} - \sin i;$$

whence

$$\sin i_1' - \sin i_2' = \frac{2\lambda}{l}.$$

But $\sin i_1' - \sin i_2' = \Delta \cos i'$, where $i' = \dfrac{i_1' + i_2'}{2}$,

as may be seen by the following:

$$\sin i_1' - \sin i_2' = 2 \cos \frac{i_1' + i_2'}{2} \sin \frac{i_1' - i_2'}{2};$$

but $\sin (i_1' - i_2') = i_1' - i_2'$ for small values of $(i_1' - i_2')$, and therefore $\sin i_1' - \sin i_2' = \Delta \cos i'$. Substituting this value for $\sin i_1' - \sin i_2'$ in the equation at the bottom of the preceding page,

$$\Delta = \frac{2 \lambda}{l \cos i'}. \tag{10}$$

Dispersion. The second of equations (7) shows that the diffraction angle i' is a function of the wave-length of the incident light. Consequently the maxima for light of different wave-lengths lie at different points, and hence white light is drawn out into a series of continuous spectra by a grating. There is one spectrum for each value of m, these spectra being called the first, second, third, etc. orders, according as m has the values $\pm 1, \pm 2, \pm 3, \cdots$. This applies only to equations (7). In equations (8), m may be said to designate orders of *minima.*

The rate of change $\left(\dfrac{di'}{d\lambda}\right)$ of the angle of diffraction with changing wave-length is known as *dispersion* (D). Differentiation of (7) gives

$$D = \frac{di'}{d\lambda} = \frac{m}{d \cos i'}. \tag{11}$$

Dispersion is therefore *inversely proportional to the grating space* and *independent of the number of lines.* For a given grating it depends only on the angle of diffraction, and for limited ranges of small values of this angle is nearly constant.

Normal spectra. For the ideal case, in which we may consider the dispersion to be constant all along the spectrum, and in which i' is therefore proportional to wave-length, we have the *normal spectrum.* Gratings, if appropriately mounted, may give spectra which approximate very closely to the normal over limited ranges of wave-length.

Overlapping of spectra. The visible spectrum lies between 4000 and 8000 angstrom units of wave-length. Substitution of these limits in equations (7) (assuming $i = 0$) shows that the corresponding values of i' for the third order, for example, lie between $\frac{.00012}{d}$ and $\frac{.00024}{d}$ radians and, for the fourth order, between $\frac{.00016}{d}$ and $\frac{.00032}{d}$ radians. So two thirds of the third-order spectrum is overlapped by a part of the fourth order. There is the same type of overlapping in other orders, though to different extents. There will be none in the first order. Fig. 34 shows the overlapping in the visible spectrum for different orders. If the spectra are photographed, in place of being visually observed, there will be a similar overlapping, but the range of wave-lengths will be quite different.

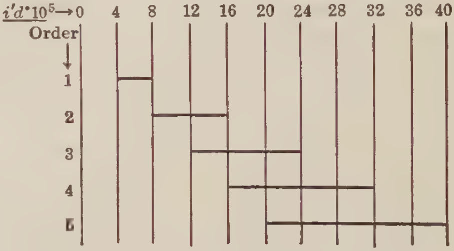

FIG. 34. Overlapping of successive orders of visible spectra

Resolving power of a grating. The resolving powers of the single and double slit have been discussed from the standpoint of their ability to form separate images of *two adjacent point sources* in monochromatic light. But the value of the angle of diffraction i' for a maximum of intensity depends not only upon the angle of incidence i, but also upon the wave-length, for the grating and the single and double slit alike. The resolving power of the *grating* is usually considered from the standpoint of its ability to form separate images of *two adjacent wave-lengths* emanating from a single source.

A monochromatic source of wave-length λ incident on a grating will give a series of maxima whose positions are determined by the last of equations (7). If there is present in the source another wave-length $\lambda + d\lambda$, these maxima will in general exist in pairs. The angle between the components of a pair may be called di', and its value is given by equation (11). The two components of a pair will be just resolved when the maximum of one component falls on a first minimum of the other (see page 42); that is, when their distance apart equals half the width of one of them, and therefore when $di' = \frac{\Delta}{2}$. Consequently the two will be separated only when

$$di' = \frac{m d\lambda}{d \cos i'} \geqq \frac{\lambda}{l \cos i'} \quad \text{or} \quad \frac{ml}{d} \geqq \frac{\lambda}{d\lambda} \quad \text{or} \quad mN \geqq \frac{\lambda}{d\lambda} \qquad (12)$$

(see equations (10) and (11)). When the two wave-lengths are *just resolved*, (12) becomes

$$\frac{\lambda}{d\lambda} = mN = R. \qquad (12')$$

The ratio $\frac{\lambda}{d\lambda}$, where $d\lambda$ is the smallest resolvable wave-length difference, constitutes the general definition of wave-length resolving power (R) of an optical instrument. By equation (12′) this is seen in the case of the grating to be *proportional only to the total number of lines* (for a given order) and hence *independent of the grating space.*

Relation between dispersion and resolving power. Under some circumstances, only a part of the grating surface is utilized. Let w be the width of that portion, N' the number of rulings in the width w, and R' the resolving power of the portion used. Then the width a' of the diffracted beam is $w \cos i'$,

or $$\cos i' = \frac{a'}{w}. \qquad (13)$$

Hence $$D = \frac{m}{d \cos i'} = \frac{N' m w}{N' d a'}.$$

But $$N'd = w \quad \text{and} \quad N'm = R';$$

and therefore $$D = \frac{R' w}{a' w}, \quad \text{so} \quad R' = a'D. \qquad (14)$$

Simple theory of slit and grating. Some of the conclusions of the last three chapters may be deduced without the amount of analysis which has been used here. For example, the positions of the minima for the single slit, of the minima and maxima for the double slit, and of the principal maxima for the grating may be accurately obtained from very simple considerations. Such developments are often termed the *simple theory* of the slit or grating respectively. This "simple theory" may be found in almost any good textbook on physics (see, for example, Millikan and Mills' "Electricity, Sound, and Light," chaps. xxi, xxii).

It requires a more thorough analysis, on the other hand, to determine the true positions of the maxima and the general law of distribution of intensity in the single-slit pattern, and to investigate the nature of the illumination at points other than the principal maxima for the grating. These points and others of a similar nature have been deemed too important to omit in any study which aims to present a point of view which is at all in advance of the merely elementary.

EXPERIMENT V

THE DIFFRACTION GRATING

Apparatus. Grating measurements are often made with the aid of a spectrometer. The grating should be so mounted that it can be placed upon the prism table of the spectrometer and be adjusted about two horizontal axes, one of which is parallel, the other perpendicular, to its surface. An excellent arrangement of leveling-screws to effect these two adjustments independently of one another is illustrated in Fig. 35. The directions for this experiment assume that the grating used is of the reflection type. They need be modified in only a few respects to apply to a transmission grating.

Adjustments. The spectrometer must be adjusted, as described in Appendix II, before the grating is mounted. The grating should then be so placed upon the spectrometer table that the axis of the instrument lies in the surface of the grating. This adjustment is not particularly exacting, and may be made by simple inspection.

There are three additional adjustments to be made:

1. *To render the surface of the grating parallel to the axis of the instrument.* This is done with the aid of the Gauss eyepiece, manipulating the screw A or B (Fig. 35) until the image of the cross hairs reflected from the grating surface is at the same level as the cross hairs themselves.

2. *To set the rulings of the grating parallel to the axis of the instrument.* To test the accuracy of this adjustment, observe whether the spectra produced by the grating are at the same height as the reflected image of the slit. This observation is facilitated by temporarily fastening across the slit a hair or fine wire, and observing the height of the shadow in the spectra produced by the grating. A continuous spectrum, such as that given by an incandescent electric lamp, is to be preferred in making this adjustment. In case it is found that the rulings are not parallel to the axis of the instrument, turn the screw C (Fig. 35) until the proper adjustment is effected.

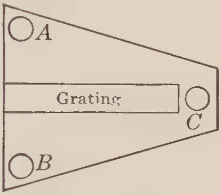

Fig. 35. Mounting for grating

3. *To set the slit parallel to the rulings on the grating.* Illuminate the slit with a source of such a nature as to produce a line spectrum; for example, the mercury arc or helium tube. Set the telescope on a prominent line other than the reflected image and render the slit as narrow as possible. Rotation of the slit about the axis of the collimator will cause a change in the definition of the spectral line under observation. Set the slit at the position which produces the maximum distinctness, taking great care, in so doing, not to disturb the adjustment of focus of the collimator.

A plane grating may be used in a number of ways in connection with a spectrometer:

1. *The grating may be set normal to the telescope*, and the collimator rotated. This simplifies the calculations, since in equations (7) $\sin i' = 0$. It involves the necessity of moving the source along with the collimator, which is usually a troublesome manipulation.

2. *The grating may be set normal to the collimator*, and the telescope rotated. In this case, in equations (7), $\sin i = 0$. It is rather troublesome, however, to set the grating exactly normal to the collimator, and the method is little used.

3. *The grating may be set at the position of minimum deviation.* In this case, as in that of the prism, $i = i'$, and equation (9″) becomes available. The result is, in the case of a reflection grating, that the diffracted rays nearly retrace the paths of the incident rays. Only one lens is necessary in this case, since the collimating lens acts also in the capacity of a telescope lens. This arrangement is quite common, being known as the *Littrow mounting.* It is illustrated in Fig. 36. In this mounting the grating is slightly tilted, so that the image of the spectrum is thrown just above or below the slit through which light is admitted to the collimator. As may be seen, this method does not adapt itself to use with the spectrometer in the case of a reflection grating, a special Littrow mounting being necessary. With a transmission grating, however, the spectrometer may be used by this method, although a rather laborious resetting of the grating and telescope is involved for each line upon which a measurement is taken.

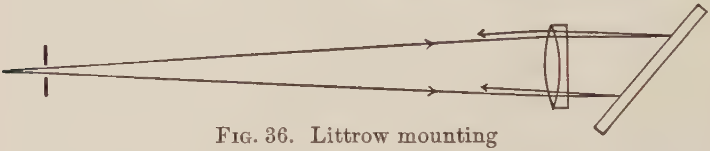
FIG. 36. Littrow mounting

4. *Telescope and collimator may be clamped at a fixed angle with one another,* and the grating rotated. This also enables some simplification to be made in equations (7). To carry out the measurements by this method, (1) set the grating normal to the telescope and read its position; (2) turn the spectrometer table carrying the grating, until the reflected image of the slit appears in the telescope, and read the new position. Call α the angle between these two positions. (3) Turn the grating still farther, until the mth-order spectrum is visible, and set the cross hairs on the desired line; call this second angle β. Consideration of Fig. 37 will show that

$$\alpha - \beta = i \quad \text{and} \quad \alpha + \beta = i'$$

and that therefore

$$\alpha = \frac{i + i'}{2} \quad \text{and} \quad \beta = \frac{i - i'}{2}.$$

We may therefore utilize equation (9). See, however, page 66 for the convention concerning the sign of i' in case of the reflection grating. This will affect the form of equation (9). It is scarcely necessary to state that this method can be used only with those spectrometers possessing a divided circle on the prism table.

5. *The grating may be set at any desired arbitrary angle*, and the telescope rotated. Equations (7) are then applied unaltered. From some standpoints this is preferable to any of the other methods, in that it makes it possible to set the grating at the angle of incidence most advantageous from the standpoint of illumination. For example, in Fig. 38 the angle of incidence is such that the beam from the collimator just covers the face of the grating. If the angle of incidence were smaller, only a portion of the face of the grating would be illuminated; if larger, only a fraction of the aperture of the collimator would be utilized. In either case a loss of efficiency would result. This method is recommended for the measurements of this experiment.

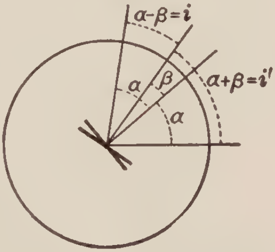

Fig. 37. Relations of angles for grating measures

Measurements. *Measurements of wave-length.* Measurements on a grating in connection with a spectrometer involve a determination of the angles of incidence i and of diffraction i' for one or more spectral lines of a known order m, and hence (see equations (7)) enable us to state a relation between the wave-length λ of the lines observed and the grating space d. This relation may be utilized either to evaluate the constant of a grating by observations on lines of known wave-length, or to determine the wave-length of spectral lines by means of a grating whose constant is known. Since the constant of a grating is usually given by the makers, the grating constitutes a ready means of wave-length measurement. It is in this way that we shall use it.

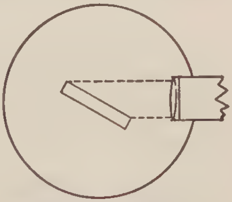

Fig. 38. Utilization of full surface of grating

Having previously set the grating to the position of best illumination, as described above in the fifth method, set the telescope normal to the grating by means of the Gauss eyepiece, as described on page 209, and read the graduated circle. This should be carefully done, since all subsequent angles are to be measured from this position of the telescope. Set the telescope on the reflected image of the slit and take the reading. The angle between this reading and the normal is the angle of incidence i.

Place the source whose radiations are to be investigated (for example, a mercury arc, helium tube, etc.) before the slit, make settings on all desired lines in every order securable, and take the appropriate readings. These readings, subtracted from the normal reading, give the corresponding values of the angle of diffraction i'. If, in making these settings with a reflection grating, the telescope is on the same side of the normal as the reflected image (direct image, in case of transmission grating), the sign of i' is negative. Otherwise i' is to be taken positive.

The value d of the grating space will be furnished by the instructor. Substitute in equations (7), to find the required values of λ.

Resolving power. Set an intense sodium flame before the slit of the collimator and render this slit as narrow as is practicable. Measure the value of i' for either of the D lines. Place an auxiliary slit, opened wide, over the objective of the telescope. Narrowing this slit will cause the individual spectral lines to *broaden*, just as in Experiment III. In this way narrow the auxiliary slit until the two components of the D line merge into one another. When they are no longer distinguishable as two separate lines, remove the auxiliary slit, and measure its width a' by means of a micrometer microscope (see page 214). The resolving power R' of that portion of the grating utilized in this observation will be given by equation (18) of Chapter VI, where the wave-length λ is the arithmetical mean of those of the two lines observed (5890 and 5896 a.u.) and $d\lambda$ is their difference. The resolving power R of the *whole grating* will be given by equations (19) and (20) of Chapter VI, which apply equally well to either grating or prism (call l the width of the ruled portion of the grating). Compare this with the value of R given by equation (12′) of this chapter. In this case, since the grating space d is known, the total number of lines N in the grating may be readily found by measuring the width l of the ruled portion.

Dispersion. Calculate D for sodium light with the help of equation (11). As a check, calculate D by equation (14), using the values of R' and a' found above.

CHAPTER VI

THE PRISM

Nature and function of prisms. An optical prism is a transparent solid possessing at least two plane polished faces which intersect. The line of their intersection is called the *edge* of the prism. The angle inclosed by the two plane surfaces is called the *refracting angle.* A plane passing through the prism parallel to the edge and perpendicular to the plane bisecting the refracting angle is called the *base*, and a plane perpendicular to the edge a *principal section.*

Prisms are usually so designed that the principal section is triangular. The base may or may not be polished. In the latter case there is no question as to which angle of the prism is the refracting angle. In the former, some care is necessary to avoid confusion, for any one of the three angles is capable of acting in that rôle. Prisms are customarily, though not necessarily, made in one of two forms: the principal section is either an equilateral triangle or an isosceles right triangle. In the latter case all three faces are invariably polished; in the former, usually only two.

Aside from their uses in certain optical instruments merely as reflectors, prisms are customarily designed for two types of use: the production of spectra and the measurement of refractive indices. The present chapter is concerned primarily with the former, though the latter is necessarily involved to a certain extent. The succeeding chapter will confine itself entirely to the latter.

Course of a ray through a principal section. In Fig. 39, *ABC* represents a principal section of a prism, with *A* as the refracting angle. The line $O'H$ is drawn parallel to the incident ray. The angles i, i', r, and r' may be either positive or negative. The first two are defined as positive when they lie on the side of the normal opposite from the refracting angle. The latter two accord in sign with that of i and i' respectively. The angle δ, being the total angle through which the incident ray is bent by its passage through the prism, is called the angle of deviation. It is the algebraic sum of the deviations produced at the two faces of the prism.

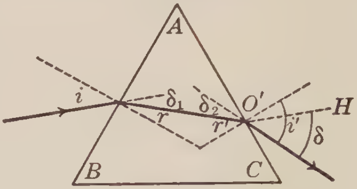

FIG. 39. Refraction through a prism

From the law of refraction,

$$\left.\begin{aligned}\sin i &= \mu \sin r,\\ \sin i' &= \mu \sin r'.\end{aligned}\right\} \qquad (1)$$

From the figure, calling the deviation at the first and second surfaces δ_1 and δ_2 respectively,

$$\delta_1 = i - r, \qquad \delta_2 = i' - r';$$

whence, adding,

$$\delta = \delta_1 + \delta_2 = i + i' - (r + r'). \qquad (2)$$

But $$r + r' = A. \qquad (3)$$

(The student may prove this.)

Therefore $$\delta = i + i' - A. \qquad (4)$$

For practical purposes it is, of course, necessary to consider a whole bundle of rays in place of a single ray. For the diverging cone of light incident on the prism from a source situated at a finite distance, there is, of course, no well-defined value of i. If, however, the light from this source is rendered parallel, all rays strike the prism at the same angle of incidence, and it thus becomes possible to apply to a wide beam the results derived for a single ray. Hence for most purposes the light incident on a prism is rendered parallel by a collimating lens.

Minimum deviation. The angle through which any ray deviates depends upon the angle of incidence. There is a certain angle of incidence, however, for which the deviation possesses a minimum value (Δ). It is convenient in many cases to use the prism in such a way that this angle of incidence is secured. In such cases the prism is commonly said to be in the position of *minimum deviation.*

The condition for this minimum may be found by the usual method of equating the first derivative to zero. That is,

$$\frac{d\delta}{di}=0 \quad \text{or} \quad \frac{di'}{di}=-1. \tag{5}$$

The second equation of (5) follows from the first by differentiating (4) with respect to i and equating to zero. An expression for $\frac{di'}{di}$ may be secured by differentiation of equations (1) and dividing, noting that, from equation (3), $dr=-dr'$. (This is left as an exercise for the student.) The result of this process, substituted in the last of equations (5), gives

$$\frac{\cos i \cos r'}{\cos i' \cos r}=1, \tag{6}$$

which is satisfied if $i=\pm i'$ (for then $r=\pm r'$). The negative sign gives the condition for a prism of zero angle, as substitution in equation (3) will show. It therefore represents a trivial case. So the conditions for minimum deviation become

$$i=i', \quad r=r'=\frac{A}{2}. \tag{7}$$

Equations (7) lead to the result that for the position of minimum deviation the incident and emergent rays are symmetrical about the plane bisecting the refracting angle, and that the once-refracted ray is parallel to the base. Fig. 40 illustrates this condition.

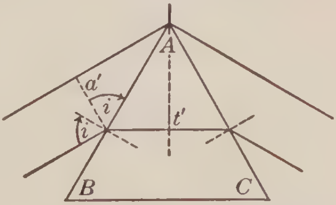

FIG. 40. Position of minimum deviation

The value of the angle of minimum deviation may be stated in terms of A and i by the appropriate substitution from (7) in (4), whence

$$\Delta = 2i - A. \tag{8}$$

Index of refraction. If the first of equations (1) is placed in the familiar form $\mu = \frac{\sin i}{\sin r}$, and the values of i and r are substituted from equations (7) and (8), there results

$$\mu = \frac{\sin \frac{1}{2}(A + \Delta)}{\sin \frac{1}{2} A}. \tag{9}$$

If, therefore, the refracting angle and angle of minimum deviation of a prism are measured, the index of refraction may be readily determined. There are other ways of measuring the refractive index of a prism. The method of minimum deviation is, however, the most convenient and the most common.

Dispersion. The value of the refractive index of any transparent body depends upon the wave-length of the light used. It is necessary, therefore, when giving a numerical value to the index, to specify the wave-length of the light used in making the measurements. The nearly monochromatic light from the sodium flame is customarily used in determinations of this sort.

It is very often necessary to specify the way in which the refractive index of a transparent substance varies with changing wave-length. The number expressing this property is sometimes quite as important as that expressing the refractive index for a given wave-length. In a prism this change $\left(\frac{d\mu}{d\lambda}\right)$ produces a corresponding change $\left(\frac{di'}{d\lambda}\right)$ in the angle of emergence for a given angle of incidence. The ratio $\left(\frac{di'}{d\lambda}\right)$ is commonly termed the *dispersion* (D), and is connected with the ratio $\left(\frac{d\mu}{d\lambda}\right)$ by the relation

$$D = \frac{di'}{d\lambda} = \frac{di'}{d\mu} \cdot \frac{d\mu}{d\lambda}. \tag{10}$$

In order to secure an expression for the dispersion in terms of measurable constants of the prism, it is necessary to find the value of $\frac{di'}{d\mu}$ and $\frac{d\mu}{d\lambda}$ in terms of these prism constants.

A value for the first ratio may be secured by differentiating equations (1) and (3), regarding i and A as constants, and combining the resulting three equations to eliminate dr and dr'. This is left as an exercise. The result is

$$\frac{di'}{d\mu}=\frac{\sin A}{\cos i' \cos r}. \tag{11}$$

For the case of minimum deviation this becomes

$$\frac{di'}{d\mu}=\frac{\sin A}{\cos i' \cos r}=\frac{2\sin\frac{A}{2}}{\cos i'}, \tag{12}$$

since for this case $r=r'$, $A=2r$. Both A and i' are measurable, and hence the value of $\frac{di'}{d\mu}$ is determined.

To find an expression for $\frac{d\mu}{d\lambda}$, it is necessary to know the value of the refractive index as a function of the wave length of the incident light. This has been the subject of a large amount of investigation. For practical purposes the largely empirical formula due to Cauchy expresses the relation between the two quantities with a fair degree of accuracy for cases of normal dispersion within the visible spectrum. In its usual form this is stated

$$\mu=A+\frac{B}{\lambda^2}, \tag{13}$$

in which A and B are, for any given substance, constants to be determined by experiment. (A has no connection with the A denoting the refracting angle of the prism.)

By differentiation of (13) we have (disregarding sign)

$$\frac{d\mu}{d\lambda}=\frac{2B}{\lambda^3}. \tag{14}$$

An expression for the dispersion is therefore given by substitution of (12) and (14) in (10); whence

$$D = \frac{di'}{d\lambda} = \frac{4\,B \sin \frac{A}{2}}{\lambda^3 \cos i'}, \tag{15}$$

which, utilizing equation (17) below, may be stated

$$D = \frac{2\,Bt}{a\lambda^3}. \tag{15'}$$

Thickness of prism. The distance which a ray travels in a prism placed in the position of minimum deviation plainly depends upon the part of the prism upon which it is incident; for example, whether this is near the base or near the refracting angle. In this sense the term "thickness of the prism" is indeterminate, and it becomes necessary so to define it that it shall possess a definite significance.

Suppose a parallel beam to be incident upon the prism of Fig. 40. Let the prism be so placed that one boundary of the beam passes through its edge. Represent the width of the beam by a', and the distance which the other boundary travels in the prism by t'. Then, since the prism is in the position of minimum deviation,

$$\sin \frac{A}{2} = \frac{\frac{t'}{2}}{\frac{a'}{\cos i'}},$$

or

$$t' = \frac{2\,a' \sin \frac{A}{2}}{\cos i'}. \tag{16}$$

If we consider t' to be the *difference between the distances* traveled in the prism by the two boundaries of the beam, it will be evident that it is a constant, *independent* of the particular portion of the face upon which the beam happens to strike, but depending on a' for its value. Let t' therefore be our definition of the term "thickness of the prism."

If, now, a' is increased until the incident beam covers the full face of the prism, and if we call this width a and the corresponding thickness t, it is evident that t is simply the dimension of the base along a principal section and that it is given by the expression

$$t = \frac{2\, a \sin \frac{A}{2}}{\cos i'}.$$

Since as a rule t can be more easily measured than a, it is of advantage to state the above equation in the form

$$a = \frac{t \cos i'}{2 \sin \frac{A}{2}}. \qquad (17)$$

Resolving power. If a beam of light incident on a prism contains all visible wave-lengths, that part of it which corresponds to each particular wave-length will, on account of the dispersion of the prism, form its own particular image of the source. These images, infinite in number, will form a band of colored light, red at one end and violet at the other, with all the colors of the spectrum between. Although the images forming this spectrum are infinitely close together, they are individually of finite width. Adjacent images therefore overlap. The less the amount of this overlapping, the greater is the purity of the resulting spectrum.

The overlapping introduces some difficulty into the separation of the images of two nearly equal wave-lengths. The term *resolving power* applied to spectrum-producing devices means the ability of an instrument to form separate images of two adjacent wave-lengths. It is defined by the equation

$$R = \frac{\lambda}{d\lambda}, \qquad (18)$$

where R is the resolving power and $d\lambda$ is the smallest wave-length increment which is separable with the instrument in question.

One way of measuring the resolving power of a prism is to place a rectangular opening of variable width over the objective lens of the telescope with which the spectrum is being regarded. As the opening is narrowed the spectral lines viewed *broaden*, for reasons discussed in Chapter III. If attention is fixed on two lines close together (for example, the D lines of sodium), these two lines will ultimately merge into one another. When this is about to occur, we are at the limit of resolution of the instrument for the aperture a' used. The numerical value of the resolving power R' will be given by equation (18).

It will be evident that under these circumstances only a fraction of the resolving power which the prism is capable of yielding is being utilized. The maximum resolving power R which is securable from the prism will be given by multiplying R' by the ratio $\frac{a}{a'}$, where a' is the width of the auxiliary slit and a is the width of the beam necessary to utilize the full thickness of the prism. That is,

$$R = \frac{aR'}{a'}. \tag{19}$$

But the value of a will be $a = l \cos i'$, where l is the length of one of the faces inclosing the refracting angle of the prism.

Therefore

$$R = \frac{l \cos i'}{a'} R', \tag{21}$$

which gives the resolving power of the whole prism for that part of the spectrum for which R' was determined.

Relation between dispersion and resolving power. It has been shown in Chapter III that the angular width of the image of a slit source produced by a rectangular aperture of width a is $\frac{2\lambda}{a}$, and that two adjacent images are in general just separable when the angular separation di' of their centers is equal to one-half the angular width of either; that is, when

$$di' = \frac{\lambda}{a}. \tag{22}$$

Dividing both sides of this equation by $d\lambda$, considering an aperture a which corresponds to the full thickness of the prism,

$$\frac{adi'}{d\lambda}=\frac{\lambda}{d\lambda}, \tag{23}$$

or, noting equations (10) and (18),

$$aD=R; \tag{24}$$

whence, combining with equation (15′),

$$R=\frac{2\,Bt}{\lambda^3}. \tag{25}$$

Equation (25) indicates that the resolving power, like the dispersion, varies directly as the thickness of the prism and inversely as the cube of the wave-length of the incident light. A comparison of (15′) and (25) shows, however, that the resolving power of a prism is independent of the angle of incidence, while the dispersion is a function of that angle. The student will justify this statement and be prepared to discuss the point.

Equation (25) may be used to check the value of R as determined by equation (21), as well as to calculate its value for portions of the spectrum to which the method of equation (21) is not applicable.

Determination of Cauchy constants. The constants A and B of the Cauchy dispersion equation (equation 13) may be determined for a given specimen of glass in various ways, depending somewhat on the form of the available specimen. In general, the readiest way is to measure the index of refraction (μ_1 and μ_2) for two different wave-lengths of light (λ_1 and λ_2), substitute these in equation (13), and solve the resulting simultaneous equations for A and B. The student may show that under these circumstances the values of A and B come out respectively

$$A=\frac{\mu_1\lambda_1^2-\mu_2\lambda_2^2}{\lambda_1^2-\lambda_2^2}, \qquad B=\frac{\lambda_1^2\lambda_2^2(\mu_2-\mu_1)}{\lambda_1^2-\lambda_2^2}. \tag{26}$$

Description of dispersive power. The dispersive power of a sample of glass is best described by giving the value of B of

Cauchy's equation or, better yet, by the values of the one or more constants involved in other classical dispersion formulas, such as Sellmeier's. This is not at present, however, the practice among practical opticians. The dispersive power is usually described by the value of ν in the equation

$$\nu = \frac{\mu_D - 1}{\mu_F - \mu_C}, \tag{27}$$

where the refractive indices are referred to the C, D, and F Fraunhofer lines ($\lambda = 6563$, 5893, 4861 a. u. respectively). The numerical value of ν varies from about 70 for light crown to about 20 for exceedingly dense flint. It will be noted that low values of ν represent high dispersive powers and vice versa.

The direct-vision spectroscope. A prism is usually mounted on a spectrometer for all measurements of the type outlined in connection with this chapter. There are some cumbersome features of this combination, however, which become considerably aggravated when, for purposes of increasing the dispersion, two or more prisms are mounted on the same spectrometer table.

The so-called *direct-vision spectroscope* utilizes the dispersive properties of a combination of several prisms but avoids most of the troublesome mechanical features of such a combination. This is due to the fact that the direct-vision spectroscope so combines prisms as to add their dispersive effects but essentially to cancel their deviations. The line of sight from the eye of the observer to the source under investigation is therefore nearly straight, and the troublesome element of providing for large deviation is eliminated.

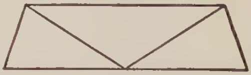

FIG. 41. Three-element direct-vision prism

The essential feature of the direct-vision spectroscope is the prism combination. This is illustrated for a three-prism system in Fig. 41. For the five-prism combination two prisms are usually of flint glass and three of crown. This type of prism combination was originally due to Amici (1860) and is consequently often termed an *Amici prism.*

An arbitrary scale, either mounted on the instrument or reflected into the field of view, facilitates the identification of spectral lines. The readings may be converted into wave-lengths by the use of a calibration curve, constructed in much the same way as the dispersion curve described in the experiments pertaining to this chapter.

EXPERIMENT VI

A. MEASUREMENT OF ANGLE OF PRISM

Apparatus. Measurements on a prism are most readily made with a spectrometer. This instrument is described and its adjustments are specified in Appendix II, p. 207. It will be assumed that some form of spectrometer is available, and that it has been adjusted by the methods described in Appendix II. There remains the process of mounting and adjusting the prism.

Adjustments. The following directions have been adapted from Chapter XV of D. C. Miller's "Laboratory Physics."

To mount the prism. The spectrometer table usually includes a prism-holder whose level is adjustable. For convenience it is desirable that the angle between the two lines joining one of the leveling-screws in the base of this prism-holder to the two other screws should be the same as the angle between the two faces of the prism. If this is the case, place the prism on its support in such a way that each of its two faces shall be perpendicular to one of these lines joining the leveling-screws. Then, as may be seen from Fig. 42, the face *AB* can be leveled by the screw *D* without disturbing the level of the face *AC*, and, similarly, for the other face.

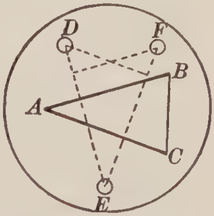

Fig. 42. Position of prism on stand

Before beginning the adjustment of the prism it will be well to turn the table and telescope into all the positions which they will later occupy in making the measurements, in order to observe whether it will be possible to read the two verniers of the divided circle for each of the settings. Sometimes one of the verniers may come under the collimator or telescope and be inaccessible. This difficulty may usually be obviated by altering the direction in which the prism stands on the table. It should also be noticed whether in these positions the prism faces are in the most advantageous relations to the objectives of the telescope and collimator. The best position is not always that in which the prism is centrally placed on the holder.

To adjust the prism. The two prism faces must be parallel to the vertical axis of the spectrometer table (or of the telescope). Arrange the Gauss eyepiece and light to observe the reflected image of the cross hairs from one face of the prism. By turning the spectrometer table, and by altering the leveling-screws in the prism support only, bring the reflected image into coincidence with the direct image. The first face is then in position. Turn the spectrometer table until the cross hairs are seen reflected from the second face of the prism, and, by the principle illustrated in Fig. 42, adjust the second face. This should not have disturbed the first face; but reobserve the image in the first face, correcting any displacement which may have occurred. Continue until the reflected image from either face coincides with the direct image, when the prism will be in adjustment.

Measurements. In all measurements be sure to record the number of the prism. This will be found etched on the top or bottom.

Reading divided circles. Both the axis of the spectrometer table and the axis about which the telescope rotates should pass exactly through the center of the divided circle. Failure to fulfill this condition causes *eccentricity.* This error is wholly eliminated from the results by reading two opposite verniers and taking the mean. If the student is unfamiliar with the use of a vernier scale, he should consult page 211.

One vernier may be designated as vernier *A* and the other as vernier *B.* To avoid confusion the simple method may be adopted of calculating the angle from the readings of vernier *A* and separately from vernier *B*, and taking the mean. If a vernier is rotated through the zero point between two readings, this must be taken into account in computing the angles. The readings for an angle reduced by this method would appear as follows:

	Vernier *A*				Vernier *B*		
First face	335	5	20		155	2	20
Second face	95	7	10		275	6	40
Difference	120	1	50		120	4	20
Angle				120 3 5			

A somewhat more concise plan is to record the degrees of vernier A only, but to use the mean of the minutes and seconds of the two verniers. The above measurements in this method would appear as follows:

	First Face		Second Face
Vernier A	335 5 20		95 7 10
Vernier B	2 20		6 40
Mean	335 3 50		95 6 55
Angle		120 3 5	

The method of averaging the entire readings of the two verniers for each setting should be avoided, as it is more cumbersome.

Angle of prism: first method — without Gauss eyepiece. Place a light L (Fig. 43) so as to illuminate the slit. Turn the prism P with its refracting edge toward the collimator C, dividing the beam of light, part falling on one face of the prism and part on the other. The exact position of the prism is not important. Clamp all movable parts of the spectrometer except the telescope. Set the telescope in the position T_1 on the image of the slit reflected from the first prism face. Read the two verniers of the divided circle as explained above. Turn the telescope to the position T_2, set it on the image of the slit reflected from the second face, and read the verniers again. The angle through which the telescope has been turned is twice the prism angle. The proof of this statement is left as an exercise.

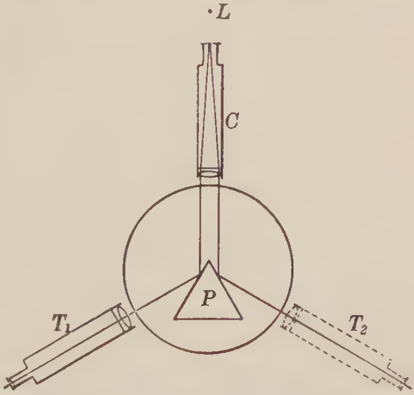

Fig. 43. Angle of prism — first method

Angle of prism: second method — with Gauss eyepiece. Clamp the telescope and arrange a light, the collimating eyepiece, and the prism so as to obtain direct and reflected images of the cross hairs as described in the directions for adjustment of the spectrometer. Secure coincidence between the two images by rotating the table only, which is possible if the adjustments have not been disturbed. Read the verniers, rotate the spectrometer table until coincidence between the direct and reflected images of the cross hairs is secured with the second prism face, and again read the verniers. The table has been turned through an angle which is the supplement of the prism angle. The proof of this statement is left as an exercise. This second method for measuring the angle between two reflecting surfaces is the more precise one.

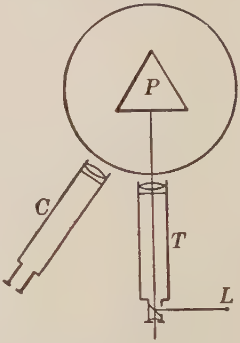

Fig. 44. Angle of prism — second method

B. INDEX OF REFRACTION BY METHOD OF MINIMUM DEVIATION

Apparatus. The same as in part *A*, except that the plane parallel mirror will no longer be needed. Sodium burner.

Adjustments. The spectrometer to be used is presumed to have been adjusted. Place the prism in the approximate position of Fig. 45. Using sodium light, find the image of the slit with the unaided eye. Rotate the prism, and find the position at which the image of the slit reverses the direction of its motion while the prism is being rotated continuously in one direction. Set the telescope on the image; and with the aid of the increased precision given thereby, locate as accurately as possible the point at which the reversal of direction of motion of the image takes place. This is the position of minimum deviation. The slit should be rendered as narrow as possible for this operation, since the error of observation increases with increasing width. The image will be found to consist of two components. It will be found preferable to take measurements on the least refrangible of these components ($\lambda = .0005896$ mm.).

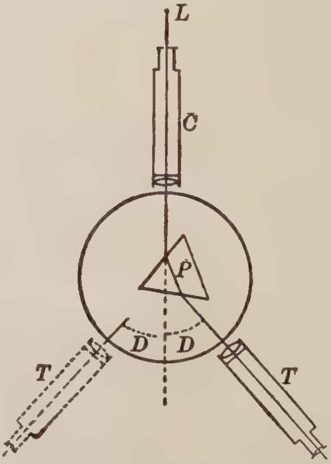

Fig. 45. Second position for minimum deviation

Measurements. Having set the intersection of the cross hairs accurately on the line in its position of minimum deviation, read the vernier. Turn the prism (not the divided circle) on the table so that it produces deviation on the opposite side, as indicated in Fig. 45, and, when the prism is adjusted for minimum deviation, set the telescope on the *same* sodium line as before. The angle between these two positions of the telescope is twice the angle of minimum deviation Δ. The refracting angle of the prism having been measured, the index of refraction of the glass of which the prism is made may now be calculated, using equation (9). This will be the refractive index for *sodium light.*

Instead of finding the deviation on the second side, it would have been possible to remove the prism after the first setting, and to sight the telescope directly on the slit through the collimator. Then the angle between the two positions of the telescope is the angle of minimum deviation. The first method is, however, preferable.

C. DISPERSION CURVE OF PRISM

Dispersion curve. The index of refraction depends on the wavelength of the light used in making the measurement. Cauchy's dispersion equation (equation 13) indicates the relation existing under certain conditions. It is the purpose of this portion of the experiment to measure μ for several values of λ distributed throughout the visible spectrum, and from the data thus supplied to determine the value of A and B of Cauchy's equation for the glass of which the prism in hand is made. When μ is plotted as a function of λ, the resulting curve is called the *dispersion curve* of the specimen in question.

Apparatus. The same as in part *B.* In addition there will be needed several sources of bright-line spectra. A mercury arc, and a helium tube operated by an induction coil or transformer, will be found, between them, to furnish a number of lines sufficient for most purposes and fairly well distributed.

Adjustments. The same as in part *B.* The prism must be set at the position of minimum deviation for each individual line. It is not correct to find the position of minimum deviation on one line, and then to take spectrometer readings on other lines without moving the prism in the meantime.

Measurements. The same as in part *B*, except that minimum-deviation readings are to be taken on all the prominent lines that are visible.

Calculate μ for each reading. Record (from a wave-length table; see Appendix I, p. 199) the wave-length corresponding to each reading. Plot values of μ as ordinates against λ as abscissas. This is the dispersion curve. Points not falling on a smooth curve indicate either erroneous readings or wrong wave-lengths.

For calculating *A* and *B* of Cauchy's equation any two pairs of values of λ and μ will give sufficient data. It is well to check these values by as many additional calculations as are possible with the data at hand. Equations (26) will be found convenient in connection with these calculations.

Determine also the value of ν (equation (27)) for the glass from which this prism is made (take values of the required indices of refraction from the dispersion curve).

The following sample set of data from readings on helium is given to aid the student in arranging his results. These results apply only to a particular specimen of glass. That the students' results do not agree with these should not therefore be a cause for misgiving.

	WAVE-LENGTHS	REFRACTIVE INDICES	INTERVAL	A	$B \times 10^6$
1.	7066	1.6467	1–6	1.624	1.078
2.	6678	1.6490	2–7	1.620	1.106
3.	5876	1.6560	3–8	1.622	1.120
4.	5048	1.6672	4–9	1.620	1.189
5.	5016	1.6676	5–10	1.626	1.198
6.	4922	1.6696		Mean **1.622**	**1.138**
7.	4713	1.6740			
8.	4471	1.6796			
9.	4388	1.6823			
10.	3888	1.6950			

D. MEASUREMENT OF RESOLVING POWER

Apparatus. The same as in part *B*. In addition an auxiliary slit of variable width, mounted on either telescope or collimator at the end next to the prism. A micrometer microscope for measurement of the width of this slit.

Adjustments. Set the prism approximately at the position of minimum deviation for sodium light. If either the prism or the aperture of the spectrometer lenses is so small that the sodium doublet is not easily resolved, the yellow doublet of the mercury arc will be found a convenient pair upon which to take the measurements for resolving power. For the sodium doublet the wavelengths are 5890 and 5896 respectively, and, for the mercury doublet, 5770 and 5790.

Measurements. Determine i'. With the auxiliary slit at full width, bring the sodium (or mercury) doublet into the field of view. Proceed as described on page 74.

Calculate R' by equation (18).
Calculate R by equation (21).
Calculate R also by equation (25).

CHAPTER VII

THE CRITICAL ANGLE

Critical angle of refraction. From the law of refraction

$$\mu = \frac{\sin i}{\sin r} \tag{1}$$

it is evident that when light is incident upon a transparent body, there will be a definite angle of refraction corresponding to every angle of incidence. If the refractive index of the body is greater than that of the medium outside, the angles of refraction will be smaller than their corresponding angles of incidence. But the upper limit of the angle of incidence is 90 degrees. There will be a corresponding upper limit for the angle of refraction which determines the largest angle which any ray penetrating the body may make with the normal. This angle is the *critical angle* of refraction r_c. Its value is given by substituting in equation (1) the condition

$$i = 90^\circ \quad \text{or} \quad \sin i = 1; \tag{2}$$

whence

$$\mu = \frac{1}{\sin r_c}. \tag{3}$$

Consequently a measurement of the critical angle of refraction gives immediate information concerning the index of refraction.

The phenomenon described by equation (3) is illustrated in Fig. 46. The incident rays will, of course, not be transmitted into the lower medium with their full intensity. Partial reflection will occur at the surface of separation of the two media. Since the representation of this fact would complicate the figure without contributing to the clarification of the principle illustrated, the reflected rays are omitted from the diagram.

To an observer at O, in a medium of refractive index μ, all rays incident upon the surface from above are confined, upon refraction, within a cone corresponding to the angles of refraction from zero up to the critical angle. No illumination exists from any direction whose angle with the normal is greater than this critical angle. Consequently the critical ray marks a boundary between a dark and a light area. For monochromatic illumination this boundary will be a sharp line. For measurements of the critical angle as commonly made in the laboratory, the circle of Fig. 47 illustrates the appearance of the field of view of the observing telescope.

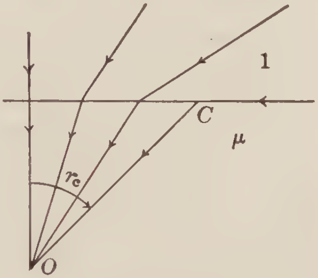

FIG. 46. Refraction at a plane surface

Index of refraction by total reflection. If the direction of the rays in Fig. 46 is reversed, so that a source is situated at O, then all rays between the normal and the critical angle will be transmitted to the upper area. It is not possible, however, for any ray making with the normal an angle greater than the critical angle to emerge into the upper medium, and hence it is reflected at its full intensity back into the lower. An observer

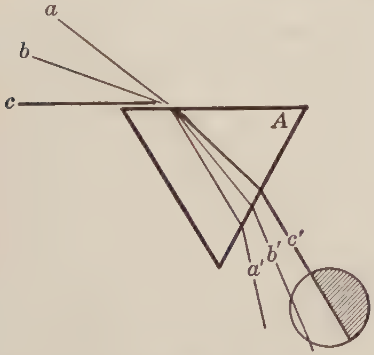

FIG. 47. Refraction at grazing incidence

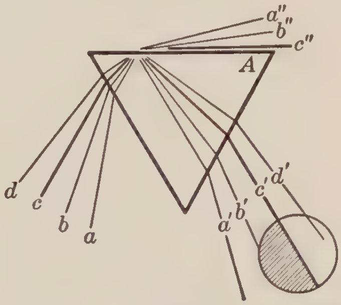

FIG. 48. Partial and total reflection

so placed as to receive these reflected rays would perceive a separation of the field of vision into two portions of unequal intensity somewhat as has been described above. In this case, however, the darker portion of the field, instead of being black, will have a considerable degree of intensity. The observer must find the line of division between the intensity due to total reflection and that due to partial reflection, instead of between the intensities due to partial transmission and zero transmission, as in the former case. These two cases are illustrated in Figs. 47 and 48 for the case of grazing incidence and total reflection, respectively, in a 60-degree prism. Since it is far easier to distinguish the line of separation in the case of Fig. 47, the method of grazing incidence is more commonly used in measurements of index of refraction.

Index of refraction by grazing incidence. Since it is not possible, as a rule, to make a direct measurement of r_c, equation (3) cannot be applied to the determination of indices of refraction. Moreover, since this method is commonly used to measure the index of refraction of a liquid by placing it in contact with a prism, it is of advantage to derive an expression to take the place of equation (3) upon the assumption that the incident beam, upon passing into the prism, passes from a medium of index μ_x into the glass of index μ_g.

Under these circumstances, for the grazing incident ray the first of equations (1) of the preceding chapter becomes

$$1=\frac{\mu_g}{\mu_x}\sin r. \tag{4}$$

The second of the same pair remains unaltered; that is,

$$\sin i'=\mu_g \sin r' \tag{5}$$

if we assume that the corresponding emergent ray passes from glass into air. In addition, we have

$$r+r'=A. \tag{6}$$

Elimination of r and r' between equations (4), (5), and (6) gives

$$\mu_x=\sin A\sqrt{\mu_g^2-\sin^2 i'}-\cos A\sin i'. \tag{7}$$

Verification of this is left as an exercise for the student. From (7) it is evident that, knowing the index of refraction of the glass of which the prism is composed, measurement of the refracting angle A and of the angle of emergence i' of the critically refracted ray gives the index μ_x of the substance covering the incident face of the prism.

The refractometer. This way of measuring refractive indices has become a standard method. Instruments for these measurements are known as *refractometers.* Several forms are on the market, differing mainly in the means taken to secure grazing incidence at the surface bounding the prism and the liquid. The working principle of one of these, the *Abbe refractometer*, is embodied in the experiments connected with this chapter.

The essential feature of this instrument lies in the use of a pair of so-called *Abbe prisms*, two similar 45-degree prisms of high refractive index. A drop of the liquid whose index is desired is placed upon the hypotenuse of one, and the hypotenuse of the other is placed upon that of the first, so that the two together form a cubical block of glass. It is then evident that for slightly convergent or divergent light incident from an appropriate direction upon the junction of the two prisms, a part will be transmitted and a part totally reflected if the index of the liquid is less than that of the prisms. A telescope focused for parallel light, placed as in Fig. 49, will show the characteristic divided field. The index of the liquid will be given, as before, by measurement of the angle between the line of division and the normal to the surface. The scales of most refractometers are so calibrated as to read the index of refraction directly.

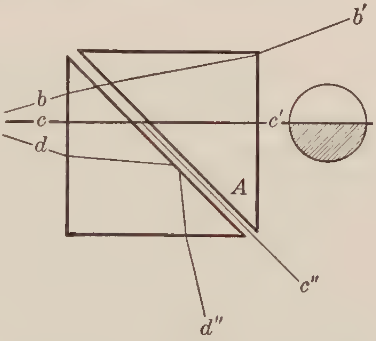

FIG. 49. Determining the refractive index of a liquid by grazing incidence

Index of Abbe prism. In order that equation (7) may be used in measuring refractive indices of liquids the index of the prism must be known. This may be determined either by the method of the preceding chapter or by grazing incidence. For the latter method substitute $\mu_x = 1$ in equation (7), from which

$$\mu_g^2 = 1 + \left[\frac{\cos A + \sin i'}{\sin A}\right]^2, \tag{8}$$

which the student is expected to verify.

EXPERIMENT VII

INDEX OF REFRACTION BY GRAZING INCIDENCE

A. Index of a Prism

Apparatus. In addition to the prism whose refractive index is to be measured, there will be needed a spectrometer, a sodium flame, and a converging lens. The latter is for the purpose of directing light from the flame upon the face of the prism.

Adjustments. The spectrometer and prism should be adjusted by the methods of the preceding chapter. After the adjustment is completed the collimator may be swung to one side or removed altogether, since it will not be needed. It is important that throughout the experiment the telescope remain focused for parallel light.

If a 60-degree prism is used, arrange prism, illumination, and telescope approximately as indicated in Fig. 47. In the case of a 45-degree prism, cause the light to be incident upon the hypotenuse. In either case note carefully the refracting angle of the prism. It will be the one bounded by the faces through which the light enters and emerges respectively. It will be found wise to shield the third face of the prism from the incident light.

Turn the telescope to such a position that the field of view is brightly illuminated by the sodium flame. Then rotate the prism and telescope, alternately, through small angles, observing the illuminated field at the same time. When a certain point is reached, the field will be seen to be sharply divided by a vertical line (which may be slightly curved) into two parts of unequal intensity of illumination, as in Fig. 47. It is upon this line of division that readings are to be made. With some practice it will be found that this line can be more quickly located with the unaided eye, after which the telescope may be swung into position.

Measurements. Record the number of the prism, which will be found etched on the top or the bottom. Set the intersection of the cross hairs on the center of the line of division and read the position. Place the telescope normal to the face of the prism, with the aid of the Gauss eyepiece. The angle between these two positions will be i' of equation (8). Some care will be necessary in deciding whether i' is positive or negative (if the emergent critical ray and the refracting edge of the prism are on opposite sides of the normal, i' is positive). The refracting angle should be measured as described in the preceding chapter.

B. INDEX OF REFRACTION OF A LIQUID

Apparatus. The apparatus is the same as in part *A*, except that the prism is replaced by a pair of Abbe prisms (described on page 87).

Adjustments. The student is cautioned to *handle the Abbe prisms with extreme care.* They are brittle and at the same time extremely soft, and are therefore easily chipped and scratched. Be sure that the surfaces are clean and that only clean liquid is used on them. Do not force them together when the liquid is between them, but merely lay one gently on the other.

Place the prisms, separated by the liquid whose index is desired, in the approximate position (relative to the incident light and the telescope) illustrated in Fig. 49. If one of the prisms was used in the measurements of the preceding section, this one should be placed on the side toward the telescope, with the measured refracting angle in the position *A* of Fig. 49. Search for the division of the field in the way described above.

Measurements. The measurements are the same as in part *B*. Determine in this way the refractive index of distilled water, and of one other liquid chosen by the instructor. The angle i' will usually be negative in these measurements. Calculate μ_x by equation (7). If the prism of the preceding section was one of the Abbe pair and is *used in the proper position,* it will be unnecessary to repeat the measurement of the refracting angle.

C. Index of Glass Plate Determined by Abbe Prism

Apparatus. Only one of the Abbe pair will be needed. The one whose refracting angle has been measured should be chosen. The glass plate whose index is to be measured replaces the missing prism. A glass specimen to be so measured should have two polished surfaces at right angles, intersecting in a sharp edge. The plate is joined to the hypotenuse of the prism by some liquid whose index is higher than that of the plate and preferably also higher than that of the prism. The use of oil of cassia for indices under 1.60, of α-monobromonaphthalene for those under 1.64, and of methylene di-iodide for those above this figure will be found satisfactory. The latter liquid should not be left uncorked, for exposure to the air causes it to crystallize and turn dark.

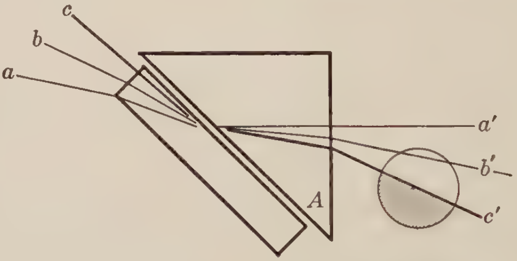

Fig. 50. Determining the refractive index of a glass plate by grazing incidence

Adjustments. Arrange as in Fig. 50. Locate the line of division. It may be that two such lines will be observed. In this case one would give the index of the liquid, the other that of the glass specimen. The one corresponding to the smaller index is the one to be chosen. In case of doubt take readings on both and reject the calculated result which gives the higher index.

Measurements. The measurements are the same as those of parts *A* and *B*. Calculate μ, using equation (7). Record the number of the glass plate.

CHAPTER VIII

INTERFERENCE BETWEEN TWO BEAMS

Conditions necessary for interference. In Chapter IV there was observed the formation of a second set of fringes, distinct from that produced by diffraction, which have been termed *interference fringes.* The interference of light from the two portions of a double slit is a special case of a general phenomenon which has played a large part in the historical development of our conceptions of the nature of light. For it was Sir Thomas Young's demonstration of the existence of interference between two beams of light that constituted the crucial experiment in the decision between the corpuscular and the undulatory theories.

It is not always easy to produce and observe interference between two beams of light. Experience shows that three conditions must be fulfilled before the phenomenon can be observed:

1. The two interfering beams must have originated in the same source.

2. Either (1) the source must be sensibly monochromatic or (2), in the case of white light, the lengths of the optical paths traveled by the two beams must be the same to within a few wave-lengths of light.

3. The interfering beams must converge at a small angle.

The first two conditions are necessary to the production, the third to the observation, of interference phenomena.

Discussion of conditions. 1. The necessity for the first condition is perhaps not immediately apparent. The common occurrence of interference (beats) in sound, between two sources of musical tone of nearly the same pitch, furnishes some ground for speculating upon the possibility of interference between two separate sources of light. Such a phenomenon has, however, never been observed, and there are reasons for believing that it never will be. In the case of musical tones there is a constancy in the phase difference between the vibrations constituting two sources of the same pitch, which to the best of our knowledge does not exist in light. It is this constancy which renders possible interference in sound. Its absence would render interference between separate sources impossible.

The probable erratic changes of phase in the vibrations constituting even a monochromatic source of light may be crudely illustrated by picturing the behavior of a swinging pendulum which is struck at irregular intervals. It is believed that an analogous condition exists in the electrons, atoms, and molecules which set up or modify the electromagnetic disturbances called light. It is consequently impossible to secure interference between two independent sources.

If, however, we divide the light from a single source into two beams, carry them over different paths, and then recombine them under appropriate circumstances, we *can* secure interference. For whatever erratic changes of phase occur in one beam will occur also in the other, and constancy of phase difference will persist.

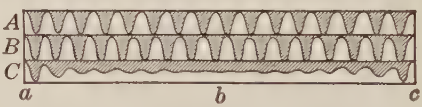

FIG. 51. Interference fringes when two wave-lengths are present

2. The necessity, in the absence of a monochromatic source, for small path difference may be shown as follows: The breadth of interference fringes has already been shown to depend upon the wave-length (equation (3), Chapter IV). If our source were composed of two wave-lengths the interference patterns, could they be separated, would appear as in Fig. 51, *A* and *B*. They are not separated, however, and hence the actual pattern would be somewhat as in Fig. 51, *C*. That is, whereas in the vicinity of the position *a* of zero path difference the fringes reënforce each other, whatever the wave-length, as we depart from this position the fringes due

to the two wave-lengths become "out of step" with each other, and partial or complete cancellation occurs (as at the point *b*). For the case cited (two wave-lengths only) reënforcement again occurs in the vicinity of the point *c*, and the cycle repeats itself a large number of times. But for white light, where all wave-lengths are present, no such secondary reënforcements will be observed, the only fringes being a few on each side of the point of zero path difference. Since cancellation occurs at different points for different pairs of wave-lengths, the fringes will be colored symmetrically on each side of the zero, or central, fringe, which is the only one perfectly achromatic.

3. The third condition is rendered necessary by the limitations of the resolving power of the human eye. When interference phenomena occur in such a way that the distance between fringes is of the order of a wave-length of light, it is impossible to observe them. An example in the field of sound exists in *stationary waves*, which are produced by interference between two waves of equal length traveling in opposite directions in the same medium.* In this case the direction between the wave normals is 180 degrees, and the distance between nodes is half the wave-length of the interfering disturbances. Under these circumstances it is, of course, possible to perceive the interference pattern. In the analogous phenomenon in light, however, the interference pattern is too "fine-grained" to be directly observable visually, though its existence has been shown by a photographic method.

If we now cause the two plane waves to converge at a small angle, the distance between points of maximum or minimum disturbance (fringes) may be made large enough to be perceived by the unaided eye. Let Fig. 52, *b*, represent such a case. Suppose the full lines to represent "crests" and the dotted lines "troughs" of the oncoming waves. Along the dashed lines ff', where crests meet crests and troughs meet troughs, there will be a maximum disturbance; whereas along the lines midway between these, where crests meet troughs, the disturbance will be a minimum. The distance between fringes is much greater than in the former case, represented by *a*, and may be increased still further by decreasing the angle at which the interfering waves converge, as illustrated in *c*. A rough idea of the minuteness of the angle required to render fringes visible to the unaided eye may be conveyed by the fact that light converging on a screen ten feet distant from two apertures a quarter of an inch apart will form fringes only a hundredth of an inch wide.

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* See Millikan and Mills' "Electricity, Sound, and Light," p. 226.

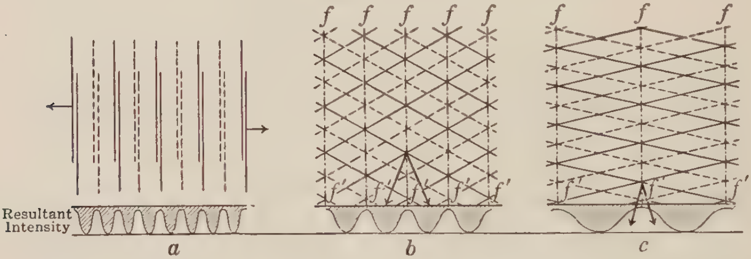

FIG. 52. Effect of magnitude of angle between interfering beams

Methods of producing interference. A rather large number of appliances have been devised for producing interference between two beams of light. Four of these are historically important. They are the double slit, the Fresnel mirrors, the Fresnel biprism, and the Michelson interferometer. The first has already been discussed (Chapter IV). The last will receive attention in the succeeding chapter. The other two will now be considered.

The Fresnel mirrors. The Fresnel mirrors were designed to meet the objections of those who regarded the double-slit experiment of Thomas Young as inconclusive. They consist essentially of two plane mirrors inclined at an angle just slightly less than 180 degrees and placed with their edges adjacent. A cone of light from an illuminated slit, which is caused to be incident upon them at a large angle, becomes converted by reflection into two cones whose axes make a small angle with each other (see Fig. 53). An eye placed anywhere in the doubly shaded area will see the two images (S_1 and S_2) of the slit source S. Interference occurs between the light which proceeds apparently from these two virtual sources.

Referring to Fig. 53, let AB represent a screen parallel to the line joining the images S_1 and S_2 and let P be the intersection of the perpendicular bisector of S_1S_2 with this screen. Let $S_1S_2 = d$, $OO' = g$, and $OP = h$. Since the only function of S is to produce the images S_1 and S_2, we may consider the illumination to come from these images as sources and ignore S.

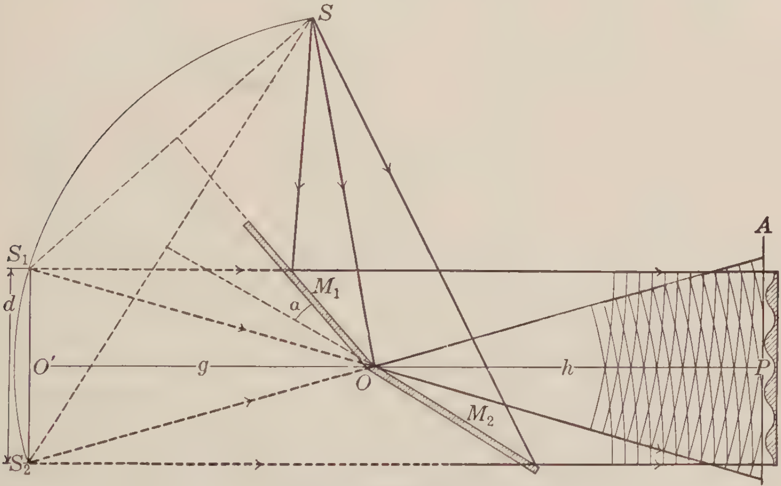

FIG. 53. Fresnel mirrors

It is required to find the illumination at any point of the screen which lies within the doubly shaded area of Fig. 53. It is quite evident that any point outside this area can receive illumination from not more than one of the slit images, and that hence at such points no interference can exist.

A little consideration will show that as thus modified the problem is identical with that of Chapter IV. Consequently the general conclusions of that chapter relative to the nature of the interference fringes observed for the double slit will apply here. One of the relations from that chapter we shall have frequent occasion to use. The angle between adjacent fringes is given by substituting successive integers for m in equation (2) or (3). Noting that in this case we are dealing with values of i' sufficiently small to justify the approximation $\sin i' = i'$, this angular width of a fringe becomes

$$i'_f = \frac{\lambda}{d}. \qquad (d = a + b \text{ of Chapter IV}) \tag{1}$$

Measurement of wave-length by the Fresnel mirrors. Before equation (1) can be applied, it will be necessary to secure a relation giving the distance d in terms of measurable constants of the Fresnel-mirror apparatus. Referring to Fig. 53, it follows as a consequence of the laws of reflection in plane mirrors that $OS_2 = OS_1 = OS$. Since these lengths equal OO' (nearly), we may regard all of them as equal to g to a sufficient degree of approximation. Therefore S, S_1, S_2, and O' lie on a circle whose center is O and which has the radius g. Moreover (from the laws of reflection), SS_2 is perpendicular to the mirror M_2, and SS_1 is perpendicular to M_1. Therefore the angle $S_1S\,S_2 = \alpha$, the external angle between the mirrors. It will be seen that S_1SS_2 is subtended by the arc S_1S_2 and has its vertex lying on the circumference of the circle, and that S_1OS_2 is subtended by the same arc, but that its vertex lies at the center of the circle. Therefore $S_1OS_2 = 2\,\alpha$.

But

$$d = 2\,g\,S_1OO' = 2\,g\alpha, \tag{2}$$

which is the relation sought.

If e is the measured width of a fringe, then the angular width is

$$\frac{e}{(g+h)} = i'_f. \tag{3}$$

Hence, substituting (2) and (3) in (1),

$$\frac{e}{(g+h)} = \frac{\lambda}{2\,g\alpha} \quad \text{or} \quad \lambda = \frac{2\,ge\alpha}{g+h}. \tag{4}$$

The wave-length may therefore be found in terms of the measurable quantities g, h, e, and α. The measurement may be further simplified by placing S at an infinite distance by means of a lens. Then g and $g + h$ become infinite together, their ratio becoming unity, and therefore

$$\lambda = 2\,e\alpha. \tag{5}$$

The Fresnel bi-prism. In place of dividing a pencil of light into two interfering beams by reflection, the same end may be secured by refraction. The Fresnel bi-prism is a device for bringing this about. It consists of a prism with one angle of nearly 180 degrees (Fig. 54). The name *bi-prism* has its origin in the fact that the device may be regarded as two prisms, of very small refracting angle, placed base to base. A cone of light incident from a source S on the base of the prism emerges as two converging cones, apparently from the virtual sources S_1 and S_2. Interference takes place between these under the same conditions as in the case of the Fresnel mirrors.

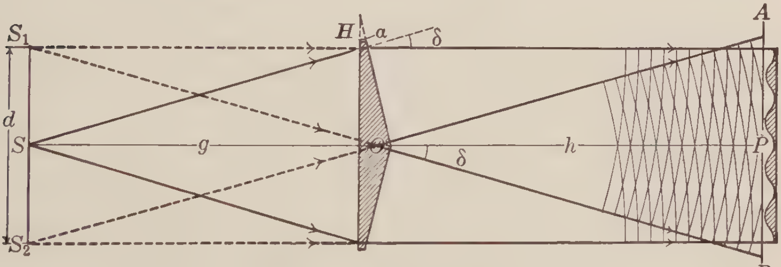

FIG. 54. Fresnel bi-prism

Measurement of wave-length by Fresnel bi-prism (divergent light). The relations by means of which the wave-length of incident monochromatic light may be determined in terms of the constants of the bi-prism and mounting are quite similar to those in the case of the mirrors. Equations (1) and (3) remain unchanged. The particular form in which d (the distance between the virtual sources) may be most advantageously expressed, however, depends upon the type of manipulation that is contemplated. If measurements are made with the source at a finite distance (divergent light), and the index of refraction μ of the glass of which the prism is composed is known, then, as will be shown,

$$d = 2\,g\alpha(\mu - 1). \tag{6}$$

To establish the truth of equation (6) refer to Fig. 54. It may be shown that the distances of the virtual sources S_1 and S_2 from the bi-prism are, to a sufficient degree of approximation, the same as that of S. All the angles involved being sufficiently small to justify considering the angles themselves instead of their sines or tangents, the following relations will be seen to hold:

$$d = 2\,gS_1HS = 2\,g\delta, \tag{7}$$

where $\overline{S_1HS} = \delta =$ the deviation produced by the prism. The value for δ may be taken from equations (2) and (3) of Chapter VI, and it should be noted

$$\mu = \frac{i}{r} = \frac{i'}{r'}, \tag{8}$$

by the law of refraction, all angles being small. By substituting in equation (2) of Chapter VI the values of $(r + r')$, and i' given by (3) of Chapter VI and (8) above, there results

$$\delta = \alpha(\mu - 1). \tag{9}$$

Substituting this value for δ in (7), equation (6) results.

We have, therefore, for the bi-prism the three equations

$$i'_f = \frac{\lambda}{d}, \tag{10}$$

$$d = 2\,g(\mu - 1)\alpha, \tag{11}$$

$$i'_f = \frac{e}{(g + h)}, \tag{12}$$

corresponding to equations (1), (2), and (3) for the mirrors. Combining to eliminate d and i'_f, and solving for λ, there results

$$\lambda = \frac{2\,ge(\mu - 1)\alpha}{g + h}, \tag{13}$$

or, in case d may be directly measured,

$$\lambda = \frac{ed}{(g + h)}. \tag{14}$$

Measurement of wave-length by Fresnel bi-prism (parallel light). The case of parallel light is most readily stated, as for the mirrors, by regarding it as a special case of divergent light (the slit being placed at an infinite distance by a collimating-lens) and making the appropriate modifications of equations (13) and (14). In (13), g and $g+h$ become infinite together, their ratio becoming unity, and therefore

$$\lambda = 2\,e(\mu - 1)\,\alpha. \tag{15}$$

In (14), substituting for d its value $2\,g\delta$ from equation (7), and modifying the resulting expression in the same way as for (13), there results

$$\lambda = 2\,e\delta. \tag{16}$$

White-light fringes. If bi-prism fringes be observed in white light, the chromatic effects in all fringes except the central white one will be somewhat accentuated in comparison with those of the mirrors on account of the dispersion produced by the prism.

EXPERIMENT VIII

A. MEASUREMENT OF WAVE-LENGTH BY A FRESNEL BI-PRISM

Apparatus. The bi-prism may be mounted either on an optical bench or on a spectrometer. The former is somewhat more satisfactory. In case an optical bench is used, the following accessories should be provided:

1. A slit provided with fine adjustments both for width and for slow rotation about an axis normal to its plane.
2. A good-quality lens, preferably achromatic, properly mounted. Its focal length must be less than one quarter of the length of the optical bench.
3. A bi-prism mounting.
4. Micrometer eyepiece.

There should be added, in case the measurements are made with parallel light,

5. An auxiliary telescope.
6. On a separate stand a reading-telescope and scale arranged in the usual manner for angle measurement. This item is advisable for use with the spectrometer as well as with the optical bench, for reasons indicated hereafter.

Whichever method of mounting is used, in addition to the foregoing there will be needed the Fresnel bi-prism, a meter stick or steel tape, a very small 45-degree prism, and intense sources of white light and of monochromatic light. A small arc, or medium-sized gas-filled incandescent with a frosted globe, is suitable for the white light, and an oxygen-illuminating-gas flame directed upon a hard-glass rod is advised for the monochromatic light.

Adjustments. *Optical-bench mounting.* When mounted on the optical bench the bi-prism may be used either with divergent light (equations (13) and (14)) or with parallel light (equations (15) and (16)). The manipulations are quite similar for the two cases. The adjustments for use with parallel light will be given, and, in case the other method is chosen, those parts (1 and 2) dealing with the adjustment of the collimating lens may be omitted. See, however, page 102, concerning proper distances between slit, prism, and eyepiece.

1. Arrange the slit, the collimating lens, and the telescope as in the plan of Fig. 55, at the same level, with the slit approximately at the principal focus of the lens. Before being mounted in position the telescope should be focused for parallel light by sighting at some distant object.

2. Illuminate the slit with the white-light source. Vary the distance between lens and slit until the image of the slit as seen in the telescope is in good focus. The lens is now rendering light from the slit parallel.

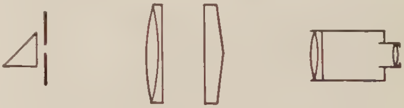

FIG. 55. Arrangement of apparatus with Fresnel bi-prism

3. Introduce the bi-prism in the position indicated. Render its base normal to the incident light in the following way: Send a beam of light through the upper half of the slit, and with the aid of the small 45-degree prism look through the lower half for the image of the slit reflected from the base of the bi-prism. If the bi-prism is rotated slowly about a vertical axis, this image will appear in the field of vision when light from the slit is reflected back along itself; that is, when the base of the prism is normal to the incident light. The slit should be made very wide until after its image is found, after which it may be narrowed down for the final adjustment.

4. Substitute a properly focused micrometer eyepiece for the telescope. If the fringes are not found to be present in its field of view, their absence may be due to one or more of the following causes:

a. The slit may not be parallel to the refracting edge of the bi-prism, in which case the slit should be rotated in its own plane by the slow-motion screw.

b. The slit may not be of proper width. It will be found of advantage to have the slit as narrow as is consistent with securing sufficient intensity.

c. The eyepiece may be at one side or the other of the interfering portion of the beam (see Fig. 54) or it may be so far away as to be outside of this portion.

When fringes appear, the field should somewhat resemble Fig. 28. Care should be taken not to mistake the diffraction fringes for the interference fringes, which are fine and sharp and harder to secure.

5. After the fringes have once been secured, their distinctness can usually be increased by altering the adjustments suggested in *a*, *b*, and *c* under 4.

Spectrometer mounting. The spectrometer lends itself readily only to the parallel-light measurements on the bi-prism. The adjustments in this case are practically identical with those for the optical bench. Substitution of the eyepiece for the telescope is accomplished by the simple expedient of removing the objective of the spectrometer telescope. Bringing the slit into parallelism with the refracting edge of the prism will probably be found the most troublesome of the adjustments, since few spectrometers are provided with the proper facilities for rotating the slit. The usual laboratory spectrometer is, moreover, scarcely accurate enough to allow of sufficient precision in measuring the small angles involved in bi-prism manipulation. The observer may make himself independent of this limitation of his instrument by attaching a mirror to the spectrometer telescope and using an auxiliary telescope and scale for angle measurements, as suggested on page 99 (in this connection see page 216).

Measurements. Directions will be given for bi-prism measurements in connection with the optical bench; the student is left to modify them for application to the spectrometer if desired.

The bi-prism will be used to measure the wave-length of the incident light. For this purpose the white light used for making adjustments should be replaced by the sodium source. In place of the few colored fringes seen with white light, the larger part of the field should now be covered by fringes (not less than twenty in number) from the monochromatic source. It will be advisable for one student to make the measurements while another maintains the oxy-gas sodium flame in adjustment. It will be wise for the latter student to wear colored spectacles for the protection of his eyes.

The measurements to be made will depend upon whether the source consists of divergent or parallel light; that is, whether equation (14) or (16) is to be used. In either case the width e of a fringe should be measured by the micrometer eyepiece, in the way described in Experiment III, p. 43. If necessary, consult page 214 for information on the use of the micrometer.

In the case of divergent light, $g + h$ may be measured with a meter stick or a steel tape, or on the bench itself. To find d, the auxiliary lens described under 2 of "Apparatus" is interposed between the eyepiece and the prism. To enable this measurement to be made, the eyepiece, prism, and slit must have been so arranged that the distance between prism and slit is not much more than the focal length of this auxiliary lens, and that the distance between eyepiece and slit is materially greater than four times the focal length.

Find a position of the lens which gives in the eyepiece clear images of the two virtual slit sources. Measure the distance d_1 between these images with the micrometer. Now find the other position of the lens for which the virtual sources are in focus. Measure the distance d_2 between *these* two images. The actual distance is given by the relation

$$d = \sqrt{d_1 d_2}. \tag{17}$$

The student will deduce this relation with the help of relations derived in Chapter II.

In case the light incident upon the prism has been rendered parallel by a collimating lens, equation (16) requires the measurement of the angle of deviation δ or, what is still easier, of the angle 2δ between the interfering beams. For this purpose a telescope carrying a mirror on its side should be substituted for the micrometer eyepiece and focused on the images of the slit. The angle 2δ through which the telescope turns in passing from one image to the other is measured with a telescope and scale in the usual manner (see page 216).

B. MEASUREMENT OF WAVE-LENGTH BY THE FRESNEL MIRRORS

Apparatus. As with the bi-prism, the mirrors may be mounted on either an optical bench or a spectrometer. The former is somewhat more satisfactory. In case an optical bench is used, the same accessories will be needed as for the bi-prism (see page 99), with the following exceptions:

1. This lens will be necessary only if the measurements are to be made in parallel light. Its focal length may have any value up to about two thirds of the length of the bench.
2. The auxiliary telescope will not be needed.
3. The small 45-degree prism will be unnecessary.

Adjustments. The adjustments of the mirrors are somewhat more troublesome than those of the bi-prism. In addition to the same manipulations as are necessary for the prism, there are several adjustments to be made on the mirrors themselves. These adjustments occasionally prove troublesome when first attempted, and it is for this reason that the bi-prism experiment should preferably precede that on the mirrors.

When mounted on the optical bench, the Fresnel mirrors may be used either with divergent light (equations (4)) or with parallel light (equation (5)). The manipulations are quite similar for the two cases. The adjustments for use with divergent light will be given, and, in case the other method is chosen, the incident light may be rendered parallel exactly as prescribed in parts 1 and 2 of the adjustments of the bi-prism.

Adjusting the mirrors. The mirrors should first be inspected, and familiarity should be acquired with the various adjustments. It will be unnecessary to describe these in detail. Provision will be found for three main adjustments:

1. Rotation of one of the mirrors about the line of intersection of the two.

2. Rotation of the same mirror about a horizontal axis in its own plane (the mirrors are invariably mounted so that their line of intersection is vertical).

3. Translatory motion of the other mirror in a direction perpendicular to its surface.

Before the mirrors are set in place on the optical bench, these adjustments should be manipulated in such a way that, as nearly as can be determined by inspection, the adjacent edges of the mirrors coincide with their line of intersection; that is, so that there is no offset, or "jump," in passing from one mirror to the other. The angle should also be set between the mirrors at very nearly (slightly less than) 180 degrees.

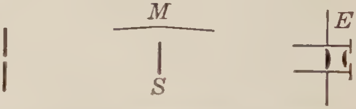

Fig. 56. Arrangement of apparatus with Fresnel mirrors

Set the mirrors in place on the optical bench, about fifty centimeters from the slit, in the approximate position shown in Fig. 56. They should be set on a level with the line from the slit to the micrometer eyepiece, but two or three centimeters to one side of this line. Adjust the slit to a vertical position, as nearly as can be determined by inspection, and place the white light before it. Rotate the mirrors about a vertical axis until an image of the slit in one or the other of the mirrors becomes visible to the eye placed about two feet to one side of the eyepiece. During this adjustment it may be necessary to remove temporarily the screen *S*, whose function is to prevent the direct light from the slit from reaching the eye of the observer.

Change the angle between the mirrors until both images of the slit (one in each mirror) become visible simultaneously and are quite close together. Place the reading-telescope before the eye, and focus it on the images of the slit. They will present somewhat the appearance of Fig. 57, *a*. Manipulate the adjusting-screws until the field appears as in Fig. 57, *b*. Move the eye (and, if necessary, the telescope) from side to side and observe whether, as each image vanishes, it disappears simultaneously along its whole length. This constitutes a test for parallelism between the slit and the line of intersection of the mirrors. If necessary rotate the slit by means of its tangent screw until this condition is satisfied.

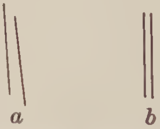

Fig. 57. Adjustment of Fresnel mirrors

In case measurements are to be made in parallel light, it is at this point that the collimating lens should be placed in position (see page 100, paragraphs 1 and 2).

Finding fringes. Replace the screen S. Remove the eyepiece from the micrometer head and, looking through its tube, rotate the mirrors about the vertical axis of their mounting until the reflected images of the slit appear in the center of the field. Replace the eyepiece. Two diffuse patches of light will be seen. Cause these to approach each other by changing the angle between the mirrors. As the patches begin to overlap, fringes should appear in the overlapping portion. In case these do not appear, make the adjustments suggested in a, b, and c under 4 on page 101 and repeat the trial. After the fringes have once been secured, their distinctness can usually be increased by altering these three adjustments, as well as by manipulating the adjustments of the mirrors themselves. It is seldom, however, that the fringes can be rendered as distinct with the mirrors as with the bi-prism.

Spectrometer mounting. The remarks under the heading "Spectrometer mounting" on page 101 may be applied, with few changes, to the case of the mirrors.

Measurements. Directions will be given for measurements with the optical bench; the student is left to modify them for application to the spectrometer if desired.

The mirrors will be used to measure the wave-length of the incident light. For this purpose the white light used for making adjustments should be replaced by the sodium source. In place of the few colored fringes seen with white light, the larger part of the field should now be covered by fringes whose number and size will depend upon the angle between the mirrors. Adjust this angle so that not less than twenty fringes are visible. It will be advisable, as in the case of the bi-prism, for one student to make the measurements, while the other, his eyes protected by colored glasses, devotes his entire attention to maintaining the oxy-gas sodium flame in adjustment.

The measurements to be made will depend upon whether the source consists of divergent or parallel light; that is, whether equations (4) or equation (5) is to be used. In either case both the width e of a fringe and the angle α between the mirrors must be determined. Measure e with the micrometer eyepiece in the way described in Experiment III (p. 43). To find α, set the reading-telescope, with its scale well illuminated, at a distance of a meter or two from the mirrors and facing them. After the adjustments necessary in angle measurement by telescope and scale (p. 216) there will be seen two overlapping images of the scale, one produced by reflection from one mirror, the other by reflection from the second. The observed amount of overlapping divided by the distance from the mirrors to the scale gives twice the angle between the mirrors, measured in radians. Substitution of the measured values of e and α in equations (4) will give the required wave-length.

In the case of measurements in divergent light, g and $g + h$ must also be measured. These may be determined by a meter stick or a steel tape, or on the bench itself, and their values substituted in equation (5).

CHAPTER IX

THE MICHELSON INTERFEROMETER

Limitations on interference applications. The fact that interference phenomena may be utilized in astronomy to augment the resolving powers of telescopes has already been indicated. They have, in fact, been applied to many problems involving the highest degree of precision. The devices heretofore described for producing interference present two serious limitations, however, which it is necessary to circumvent before interference measurements can be made applicable to a very wide range of fields.

The difficulty of securing sufficient intensity for observations under any but the most favorable circumstances is the first limitation. This has its origin in the use of one or more slits which are integral parts of the interference devices already described. These slits cause the loss of all but a minute fraction of the intensity actually existing in the source; but for the production of interference in the double slit, as well as in the Fresnel mirrors and bi-prisms, it is necessary not only that these slit sources be used but that they be exceedingly narrow. It should be noticed, however, that a narrow source is not one of the three essential conditions to the production of observable interference phenomena.

The proximity of the paths described by the rays between initial separation and final recombination in the devices already considered constitutes the second limitation. This proximity has its origin in the necessity for a small angle between the interfering rays at recombination, but, as may be observed, the proximity is not necessary to secure a sufficient minuteness of this angle. Some applications involve the introduction of rather bulky optical appliances into the path of one of the interfering rays without affecting that of the other. Proximity of the two paths makes this a difficult undertaking, as may readily be seen.

The interferometer. The first really successful instrument to surmount these difficulties was designed by Professor A. A. Michelson and was appropriately termed the *interferometer.* It has appeared in a great many forms, depending mainly on the use for which it was intended; but the form which is the most common and of which all other forms may be said to be modifications, is illustrated in Fig. 58 and presented in diagram in Fig. 59.

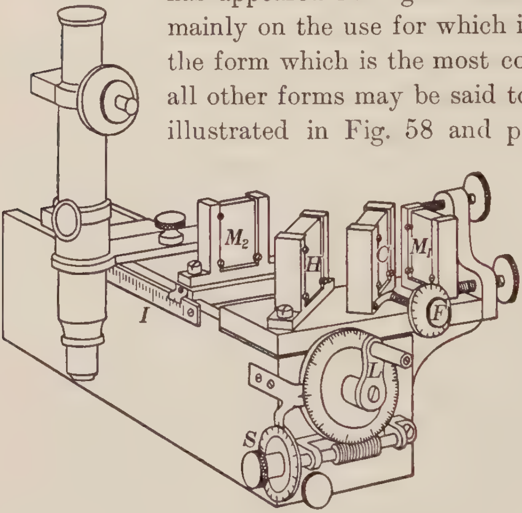

FIG. 58. Interferometer

The following description is taken from the second chapter of "Light Waves and their Uses," by Professor Michelson.*

The light starts from the source S and separates at the front of the plate H, part of it being reflected through C to the plane mirror M_1, returning exactly on its path through H and thence to O. The other part of the ray goes through the glass plate H, is reflected by the plane mirror M_2, and returns on its path to H, where it is reflected so as nearly to coincide with the first ray. The plane-parallel glass C is introduced to compensate for the extra thickness of glass which the second portion of the ray has traversed in passing twice through the plate H. Without it the two paths would not be optically identical because the first would contain more glass than the second.

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* By courtesy of Professor Michelson and The University of Chicago Press.

Some light is reflected from the rear surface of the plate H, but its effect may be rendered insignificant by covering the front surface of H with a coating of silver of such thickness that about equal portions of the incident light are reflected and transmitted.

The plane-parallel plates are worked originally in a single piece, which is afterward cut in two. The two pieces are placed parallel to each other, thus insuring exact equality in the two optical paths HM_1 and HM_2.

The foregoing principles are applied in concrete form in the instrument shown in Fig. 58. A rigid casting serves as the bed of the instrument. One end of this bed has fastened to it a heavy metal plate which carries the three glass plates M_1, C, and H. The plate H is held in a metal frame which is rigidly fastened to the base. The frame which holds C can be turned slightly about a vertical axis to allow of adjusting C so that it is parallel to H. The mirror M_1 is held by springs against three adjusting screws which are set in a vertical plate attached to the end of the base. Both M_1 and M_2 are silvered on their front faces. The frame which holds the mirror M_2 is firmly mounted on a metal slide which can be moved by the screw L along the ways. One very essential feature of the apparatus is that these ways shall be so true that the mirror M_2 shall remain parallel to itself as it is moved along. The accuracy of the ways must be so great that the largest angle through which the mirror M_2 turns in passing along them is less than one second of arc.

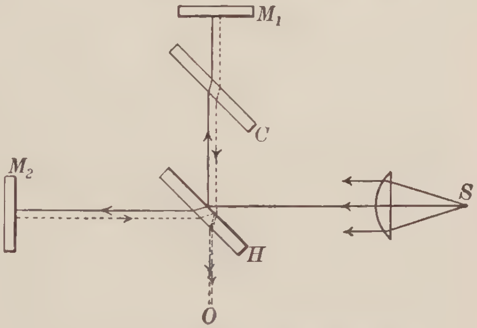

FIG. 59. Plan of interferometer

It will be observed that the two difficulties incident to some of the other forms of interference apparatus are not met in the interferometer. The initial division is performed by reflection at a half-silvered plate instead of by diffraction through a slit, with a consequent gain in intensity of illumination which is enormous. This possesses the additional advantage of eliminating the single-slit diffraction pattern, which, as has been observed, is superposed upon the interference fringes of the double slit. It is consequently possible, when using the interferometer with monochromatic light, to observe many thousands of fringes instead of the small number visible in the double-slit pattern. Moreover, the two rays which later produce interference pursue paths at 90 degrees with one another, and hence the troublesome feature of proximity of paths is avoided.

The observer may look into the apparatus from O (Fig. 58). He will see the plate M_1 directly and an image of the plate M_2 reflected by the plate H. This image will appear in the direction of the plate M_1, as far behind H as the mirror M_2 is really in front of it. The instrument is thus seen to consist essentially of a film of air inclosed between the plate M_1 and the virtual image of M_2. This feature possesses the important advantage of enabling the observer to secure interference between faces of a film as near to zero thickness as he desires, or to cause one face actually to pass through the other (the image of M_2 to pass through M_1), a feature which at times is of great utility. Moreover, in discussing the interference phenomena produced by this instrument, it is necessary to consider only this film of air bounded by one mirror and by the image of the other. We shall accordingly refer in the future to the two mirrors, the angle between them, their distance apart, etc., meaning always the mirror M_1 and the *image of M_2 in H.*

The theoretical part of the discussion of the aspects of interferometer performance that follows in the succeeding two sections is adapted from Michelson's treatment.*

Production of interference fringes. If a source of light is sufficiently broad, the illumination at any external point will be independent of its distance, form, or position. Experience justifies the application of this principle to the interferometer to the extent of postulating that the source of illumination, instead of being at S, is situated at the mirrors themselves. Therefore let OM_1 (Fig. 60) represent the mirror M_1 of Figs. 58 and 59, OM_2 the mirror M_2, and OM_1' the image of M_1 in M_2. If in accordance with our supposition above we imagine a plane, luminous surface to be coincident with OM_1, then its image in OM_1 will be OM_1 and in OM_2 will be OM_1'. For every point p of OM_1 there will be a corresponding point p' in OM_1', symmetrically placed with reference to OM_2 and possessing a phase of vibration which, though not necessarily the same as that of OM_1, differs by a constant amount. If we now suppress the actual source of light and the mirrors, and replace them by the images, the effect at any point P (not necessarily in the plane of the figure) will be unaltered.

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* *Philosophical Magazine* (Fifth Series), Vol. XIII (1882), p. 236.

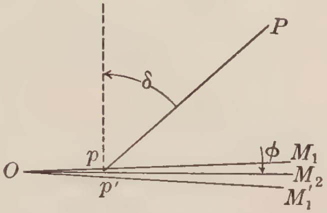

FIG. 60. Assumptions leading to analysis of interferometer action

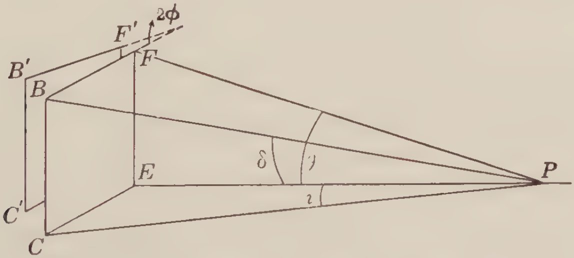

FIG. 61. Angle notation in interferometer analysis

Let δ be the angle formed by the line joining P and p (or p') with the normal to the surface OM_2, δ and ϕ being both supposed small. The difference of optical path Δ will be

$$\begin{aligned}\Delta &= Pp' - Pp = pp' \cos\delta \text{ (nearly)} \\ &= 2\,t\cos\delta, \qquad (1)\end{aligned}$$

where t is the distance between the surfaces at p.

Let $BCEF$ and $B'C'E'F'$ (Fig. 61) represent portions of the two images, their intersection being parallel to FE and their inclination 2ϕ. Suppose PE to be normal to the front surface, and let $BPE = \delta$, $CPE = i$, $FPE = \theta$. Call the distance between the surfaces at E, $2t_0$. Then, if $2t$ is the separation at any other point B,

$$2t = 2t_0 + CE \tan 2\phi,$$

or, approximately,

$$t = t_0 + CE \tan\phi = t_0 + D \tan\phi \tan i,$$

where

$$D = PE.$$

Whence, substituting in (1),

$$\Delta = 2(t_0 + D \tan\phi \tan i)\cos\delta. \tag{2}$$

But

$$\cos\delta = \frac{PE}{PB} = \frac{PE}{\sqrt{PE^2 + EC^2 + CB^2}}$$

$$= \frac{1}{\sqrt{1 + \tan^2 i + \tan^2\theta}},$$

and therefore

$$\Delta = 2\,\frac{t_0 + D\tan\phi\tan i}{\sqrt{1 + \tan^2 i + \tan^2\theta}}. \tag{3}$$

Now the production of interference phenomena depends upon the presence of an ordered sequence of phase differences in the interfering beams and usually assumes that all parts of the wave front constituting each beam are in the same phase. The phenomena are most distinct when this condition is satisfied, but they may be perceived as long as there is a predominating phase. If, however, the beams are made up of rays possessing a wide range of phases, interference phenomena cannot be observed. An examination of equation (3) will show that unless rather stringent limitations are placed upon the magnitude of the angles involved, Δ will possess a wide range of values and that hence all interference will be obliterated. If, however, we limit the angles to such values that the range of Δ is less than half a wave length of light, a little consideration will show that the range of phases of vibration present at P is so limited that there will exist a mean, or predominating, phase and that hence interference can be produced. Fortunately for convenience in use of the interferometer, the small angular magnitude of the cone of light which can enter the human eye furnishes sufficient limitation on ϕ, i, and θ for many purposes. Distinctness of the fringes may sometimes be improved by sighting through a small aperture such as a pinhole.

Position of fringes. The distance of P (from the luminous images) at which fringes are most distinct will be, as appears from the foregoing paragraph, that distance for which Δ has its minimum value. But the value of Δ depends upon the two independent variables θ and i, and hence the conditions for a minimum will be $\frac{\delta\Delta}{\delta\theta} = 0$ and $\frac{\delta\Delta}{\delta i} = 0$. If the differentiations are carried out, it may be seen that the first condition leads to $\theta = 0$ and the second to

$$D = \frac{t_0 \tan i}{\tan \phi}. \tag{4}$$

The student is expected to verify these results.

Analysis of (4) shows that if $t_0 = 0$ (that is, if the mirrors are in contact), the fringes are formed at their surface. If $\phi = 0$ (that is, if the mirrors are parallel), the fringes are at infinity. If ϕ and t_0 have simultaneously the value zero, the position of the fringes is indeterminate. If ϕ and i have the same sign, the fringes lie in front of the mirrors; if the opposite sign, behind.

Form of fringes. In practice there are two ways in which the interferometer may be used: either (1) the mirrors are maintained parallel or (2) they are held at a very small angle with one another. In the first case the fringes appear as concentric circles which, with increasing separation of the mirrors, appear to spring successively into being at the common center, and to expand as others appear within them. In accordance with the conclusions of the preceding paragraph these circles are situated at infinity, their angular magnitude being determined by the path difference of the interfering beams.

In the second case the fringes are in general curved. One of them, which corresponds to zero separation of the mirrors, is perfectly straight and, in white light, achromatic. This is commonly termed the *central fringe.* All others are curved, convex toward the central fringe. Their exact form seems to be a matter of considerable doubt, but it is not far from circular. These fringes move across the field of vision with changes of position of the mirrors. With monochromatic illumination there may be many thousands of them; in extreme cases several million. With white light their number is limited to approximately a dozen in the neighborhood of the central fringe. These white-light fringes show color effects which, if the interferometer is in perfect adjustment, are symmetrical with respect to the central fringe. In general appearance they are quite similar to those seen in the double slit or the Fresnel mirrors or prism, except that in the interferometer the central fringe is black instead of white. If the paths of the interfering beams possess the proper kind and degree of optical dissymmetry, these white-light fringes become more numerous and simultaneously more nearly achromatic.

A rigorous *theoretical* investigation of the form of the fringes seems to be a matter of considerable complexity; in fact, the general problem has never been completely solved. Some very thorough work has been done by Feussner * on interference in thin films, of which this is readily seen to be a special case. Unfortunately his results seem not to account for the phenomena observed in connection with the interferometer. On the other hand, while Michelson's theory † accounts for these phenomena, it was designed to apply to the special case in hand, and not intended as a solution of the general problem. Most of the controversy concerning the form of the fringes centers about the second case (mirrors inclined). The theory for parallel mirrors may be made comparatively simple.

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* *Annalen der Physik und Chemie* (1881), Bd. 12, S. 558, or Winkelmann, Handbuch der Physik (edition of 1906), Bd. 6, S. 941.

† *Philosophical Magazine* (Fifth Series), Vol. XIII (1882), p. 236.

For this case $\phi = 0$ in equation (2), and hence $\Delta = 2\,t_0 \cos\delta$; and if we call Δ_0 the path difference for normal incidence ($\delta = 0$ in Fig. 60), then

$$\Delta_0 - \Delta = 2\,t_0(1 - \cos\delta) = 2\,t_0\,2\sin^2\frac{\delta}{2} = \delta^2 t_0 \text{ (nearly)};$$

or, expressed in terms of wave-lengths,

$$\frac{\Delta_0 - \Delta}{\lambda} = \frac{\delta^2 t_0}{\lambda},$$

and, for an integral number n of wave-lengths,

$$\frac{\Delta_0 - \Delta}{\lambda} = n,$$

and therefore

$$\delta = \sqrt{\frac{n\lambda}{t_0}}. \qquad (5)$$

There is no term in equation (5) specifying direction; and consequently the phenomena are the same, whatever direction from the reference line is chosen. That is to say, the fringes are circular. Moreover, since the reference line is the normal to the mirrors, the common center of these circles lies on this normal. Equation (5) gives the angular diameter of these circles as a function of order, wave-length, and separation of mirrors.

Measurement of distance and wave-length by the interferometer. The interferometer may be used to measure the wave-length of monochromatic light. In experienced hands this measurement may be made with very great precision. The interferometer measurement, by Professor Michelson, of the length of the world's standard meter at Paris, in terms of the wave-lengths of three radiations of cadmium vapor, ranks as the most accurate measurement of a length ever made. One of the experiments at the close of this chapter utilizes the property of the interferometer which made this measurement possible. It takes advantage of the precision in our knowledge of wave-lengths to calibrate a *standard scale.*

Measurements of this type consist in counting the number of fringes which pass as the path difference between the interfering beams is altered. The change in path difference is effected by moving the mirror M_2 (Fig. 58) by rotating the handwheel L. The passing of a predetermined point by one fringe indicates that M_2 has been moved a distance equal to half the wave-length of the light that is being observed. The relation thus established may be utilized in either of two ways:

If careful measurement is made of the distance that M_2 moves as a counted number of fringes pass, a determination of the wave-length is made, of the same order of accuracy.

In general, however, there will be available more precise information concerning the wave-length than this method is capable of yielding in inexperienced hands. We may take advantage of this fact, and cause the fringe count to give the distance moved by M_2 with a far higher degree of precision than ordinary measurement with a standard scale would give. In fact, we may in this way calibrate such a scale.

Measurement of index of refraction by the interferometer. The interferometer may be used to determine the index of refraction of a sample of glass. Two pieces of the sample, ground and polished to optical plane-parallelism, are used. One is placed in front of mirror M_2 (Fig. 59), parallel with it. The other is placed in front of M_1, in a frame fitted with a pivot and tangent screw, so that it may be slowly rotated.

It is evident that rotating these plates is equivalent to lengthening the optical path of the rays traversing them. A mirror may be fitted into the frame which holds either of these plates so that the angle through which it turns may be measured with a telescope and scale. Fig. 62 represents one of these plates of glass, both in its normal position (parallel to the mirror) and after it has been turned through a measured angle.

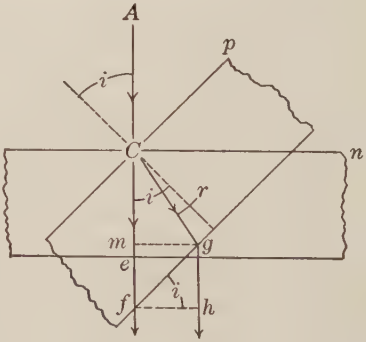

FIG. 62. Measurement of refractive index by interferometer

In its normal position the ray of light takes a straight-line path *Af*, and its angle of incidence is zero. After the plate has been turned, the ray takes the path *ACgh*, and its angle of incidence, *i*, is the angle through which the plate has been turned. The number of fringes $2n$ which pass while the plate is being turned are counted, and the problem is to find an expression for the index of refraction μ in terms of n, λ, i, and t (t is the measured thickness of the plate, and n stands for half the number of fringes counted, since the light passes twice through the plate).

Before the plate is turned, a ray traveling from C to the plane of *fh* passes through a thickness *Ce* of glass and *ef* of air; after it is turned, the ray passes through a thickness *Cg* of glass and *gh* of air. Subtraction of the optical path in the first case from that in the second case will give the path difference. But this path difference is given also by the expression $n\lambda$.

$$\text{Therefore}\qquad (\mu\overline{Cg}+\overline{gh})-(\mu\overline{Ce}+\overline{ef})=n\lambda. \tag{6}$$

$$\text{But}\quad \left.\begin{aligned} Cg &= \frac{t}{\cos r}.\\ gh &= fh\tan i = Cg\sin(i-r)\tan i\\ &= \frac{t\sin(i-r)\tan i}{\cos r} = t\sin i\,(\tan i-\tan r).\\ Ce &= t.\\ ef &= \frac{t}{\cos i}-t = t(\sec i-1). \end{aligned}\right\} \tag{7}$$

Substitution from equations (7) in equation (6) gives, after reduction, neglecting the term involving $n\lambda^2$, the expression

$$\mu=\frac{(t-n\lambda)(1-\cos i)}{t(1-\cos i)-n\lambda}. \tag{8}$$

The student is expected to verify this equation.

The interferometer in spectroscopy. No discussion of the interferometer would be complete without a reference to its contributions to the development of spectroscopy. To attempt to give the theory of this phase of its performance, or even to outline the bare facts of its accomplishments along this line, would carry us far beyond the scope of this text. We may, however, briefly sketch the nature of one type of spectroscopic problem which the interferometer has helped to solve.

We have already observed, in connection with Experiment VI, that the spectrum of an incandescent gas consists of bright lines and that the width of these lines depends upon the constants of the prism and spectrometer. It has been learned in comparatively recent times that these lines possess characteristic intrinsic widths which may be and usually are completely obscured by the much greater, though spurious, width impressed upon them by the diffraction phenomena attendant on passage of light through the usual type of spectrometer.

It has also been discovered that lines which appear single in an ordinary spectrometer are often really composed of several, sometimes many, component lines which are so near together that they are not resolved in any ordinary method of observation. The intrinsic widths of spectral lines, and the number, separation, and widths of their components, have acquired within recent years a place of very great importance in the study of the structure of the atoms which emit these radiations. It was at a time when there existed no other spectroscopic appliance sufficiently powerful to be of any use in this field, that the Michelson interferometer gave the first impetus to the experimental study of the "fine structure" of spectral lines.

Visibility curves. The width of fringes produced in interference phenomena was shown in Chapter IV to depend, among other things, on the wave-length of the light used. It was further pointed out, in Chapter VIII, that in case of the presence of two or more wave-lengths a continuous change in the path difference of the interfering beams would be accompanied by rhythmic changes in the contrast between the light and dark portions of the field; that is, rhythmic changes in the *visibility.* Graphical representations of these visibilities, plotted as functions of the path difference, have become known as *visibility curves.*

Confining attention for the moment to the purely ideal case of rigorously monochromatic sources, the visibility produced by one such source would be constant (p. 47), and the visibility curve would therefore be a horizontal straight line (Fig. 63, *a*). For two such sources, of nearly equal wave-length but unequal intensity, the visibility curve is represented in Fig. 63, *b*. Had the intensities been equal, the minima of this curve would have touched the horizontal axis.

But for the practical case even a "monochromatic" source possesses an infinite number of wave lengths, since, as has just been remarked, every spectral line is of finite intrinsic width. The effect of this finite width is a falling-off of the visibility, with increasing path difference; so that, practically, a "monochromatic" light produces a visibility curve of the type shown in Fig. 63, *c*, while for a source possessing two very nearly equal wave-lengths the curve would be as shown in Fig. 63, *d*. Similar curves could be drawn for different widths, different numbers of components, different separations, etc. Hence, by examination of a visibility curve, we may infer the nature of the source under investigation.

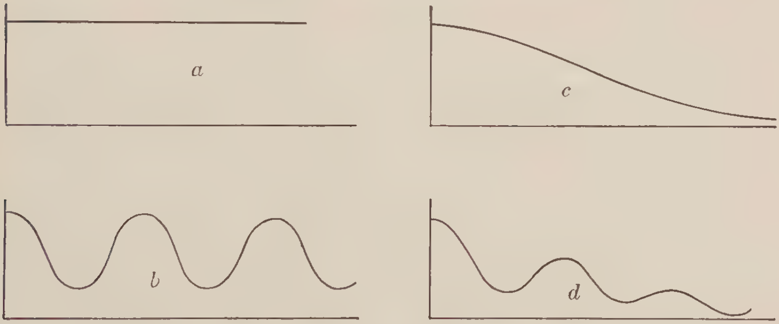

FIG. 63. Characteristic visibility curves

The Michelson interferometer has furnished the only effective means of studying visibility curves. It is necessary only to move the mirror M_2 (Fig. 59) continuously, estimating at stated intervals the visibility of the fringes which pass. It was, in fact, for this identical type of investigation that the interferometer was designed and first used. The first spectral line to be analyzed was the alpha line of hydrogen (Hα, $\lambda = 6563$), which was found to be a doublet whose components were separated by 0.14 a. u. The visibility curve for this line, as actually observed by Professor Michelson in 1882, is shown in Fig. 64, and to the left of the visibility curve the structure of the line which it revealed.

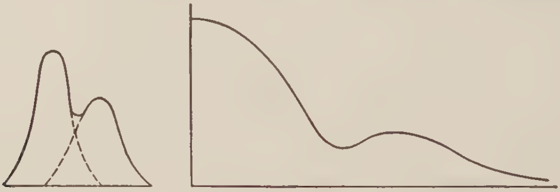

FIG. 64. Structure of Hα with corresponding visibility curve

It was not until a number of years later that gratings were made of sufficient resolving power to make possible a direct visual verification of the structure of this line.

The interferometer has since been applied to the solution of many problems of this nature, and even today it excels for some purposes some direct measurements that it is possible to make with modern spectroscopic instruments.

EXPERIMENT IX

A. CALIBRATION OF A TENTH-MILLIMETER SCALE BY THE INTERFEROMETER

Apparatus. In addition to an interferometer of the type described on page 108, there will be needed a source of monochromatic light, a converging lens, a tenth-millimeter scale etched on glass, and a micrometer microscope, the latter rigidly attached to the base of the interferometer. A very convenient monochromatic source is a mercury arc screened by an appropriate color filter so that only illumination from the green line ($\lambda = 5461$) reaches the eye. In the absence of this type of illumination an ordinary sodium burner will be found reasonably satisfactory. The lens, while often not necessary, may be so placed as to render the field uniformly illuminated, which increases the ease of observation. A hand counter, to record the number of fringes passing, will be found very convenient.

Adjustments. The student should take every precaution to *avoid touching the silvered surfaces* of the interferometer plates. The slightest contact irreparably mars the surfaces and necessitates their resilvering. If any manipulation appears to entail a possibility of accidentally touching them, the instructor should be called into consultation.

Adjusting for fringes in monochromatic light. With the plates set as described on page 109, measure roughly the distance from a chosen point of the silvered surface of plate H (Fig. 59) to the front of mirror M_1. Set the mirror M_2 (by turning the handwheel L) so that its distance from the chosen point of the silvered surface of H is roughly the same. Any attempt at accuracy in this adjustment will be a waste of time, and will involve a needless risk of touching the silvered surfaces.

It is sometimes desirable, for convenience of manipulation, for the observer to be stationed at S (Fig. 59). This change is perfectly possible. While the following directions assume the observer to be stationed at O and the light-source at S, the two may be interchanged without affecting in any way the performance of the instrument.

Place the monochromatic source in position. Hold some pointed object, such as a pin or the point of a knife blade, between the source and H. A pin may be very conveniently fastened, point up, to the frame of the interferometer with a pellet of soft wax. Upon looking into the instrument from O (Fig. 59), three silhouette images of this object will be seen. One is formed by light that is reflected at the *silvered* surface of H and that thence travels the path HM_1HO (Fig. 59). The second is formed by light which, passing through the plate H, follows the path HM_2HO, the second reflection (at H) taking place from the *silvered* face. The third is formed by light which, having been reflected at M_2, is reflected to O by the *unsilvered* surface of H. This will be less distinct than the other two. If the silvered coating on H is sufficiently thin or completely absent, a fourth image may be seen, formed by light which pursues the same path as that of the first, except that the first reflection takes place at the *unsilvered* face of H (that surface facing M_2). Usually the silver coating is so thick as to make this image imperceptible, since this portion of the light must pass through the coating three times.

Interference fringes are found by bringing the first image into coincidence with the second, both vertically and horizontally, by manipulating the adjusting-screws back of mirror M_1. As the instrument has been here described, the third image is less distinct than the other two, and the first is the only one that moves as the mirror M_1 is turned. Bringing the first and second images into coincidence causes the image of M_1 in the silvered face of H to be nearly parallel to the mirror M_2. Thus the condition described on page 111 is brought about. It is quite probable that the fringes will not appear when the two images of the point seem to have been brought into coincidence. This is due simply to the fact that the eye cannot judge with sufficient accuracy for this purpose as to when the two are really superposed. It may be necessary to cause the two images to move back and forth over each other several times before exact coincidence is secured, at which time the fringes will appear.

When fringes have been once found, their position, size, form, and distinctness may be varied by careful manipulation of the adjusting-screws of mirror M_1 and the handwheel L. The student should practice making the adjustments involved in finding and controlling the fringes, until he is completely master of the process. When this point has been reached, not more than one or two minutes will be required to find fringes after the monochromatic source has been set in place.

Adjusting for circular fringes. In general the form of the fringes observed in the preceding adjustment will be roughly that of concentric circular arcs. It will be found possible, by carefully manipulating the adjusting-screws of mirror M_1, to bring the center of these circles into the field of view, thus producing fringes which are complete circles. As is indicated on page 113, circular fringes are produced when the image of M_2 in H is parallel to mirror M_1. It does not follow, however, that when fringes that are apparently circular have been secured in the manner described above, the mirrors are rigorously parallel. In fact, the opposite is almost certain to be true. Having secured these apparently circular fringes, the best test for parallelism of the mirrors is to move the eye sideways and up and down while looking at the fringes. If the adjustment is correct, the circles will not change in diameter as the eye is thus moved. To be sure, they will appear to move across the field, because their center is at the foot of the perpendicular from the eye to the mirror (see page 115), but their diameters will remain constant as the eye is moved laterally.

This adjustment is somewhat troublesome to make, and still more so to maintain while moving the mirror M_2. This is due to the fact that any slight irregularity in the ways, if greater than a small fraction of the length of a light wave, is sufficient to destroy the adjustment for parallelism. It is, in fact, only the very best interferometers that are made with sufficient care to admit of a maintenance of parallelism for any length of time. Fortunately, rigorous parallelism is not necessary in the measurements to be made in this experiment, though it is advisable to approximate it as closely as conditions will permit.

Measurements. The measurements will consist of a calibration of a tenth-millimeter scale in the manner described on page 116. The scale is attached to the moving carriage of mirror M_2, and is observed with the micrometer microscope attached to the base of the interferometer. The greatest source of error in these observations will be that involved in setting the cross hair of the micrometer microscope on the division marks of the scale. To minimize this error it is best to abandon all attempts at setting the cross hair on the division marks, and to start the fringe count when the cross hair is somewhere between the marks. Its exact position may be determined in the following way:

There will be seen in the field of view of the microscope not only the main, stationary cross hair, but an auxiliary cross hair that moves across the field as the micrometer screw is turned. Carefully set this cross hair on either of the division marks adjacent to the stationary cross hair, and read the position of the graduated head of the micrometer screw. Displace the movable cross hair, bring it again to the division mark, and reread the micrometer screw. Repeat for at least ten settings, bringing the movable cross hair up to the division mark *always from the same side.* Record the mean of these ten readings. Run the movable cross hair over to the position of the stationary cross hair, noting the number of times that the zero mark of the micrometer screw passes the fiducial point. Make a series of ten settings of the movable cross hair on the stationary cross hair, similar in every way to the series just made on the division mark, and record the mean. The distance from the division mark to the stationary cross hair, in terms of number of turns of the micrometer screw, then becomes quite accurately known. Suppose, for example, that the mean of the first ten settings was 36.43, of the last ten 72.55, and that from one position of the movable cross hair to the other the micrometer screw passed the zero mark twice in the direction of increasing magnitude of graduation figures, there being one hundred divisions on the circumference of the micrometer screw. The required distance is then $.6357 + 1 + .7255 = 2.3612$.

In the same way as before, run the movable cross hair over to the division mark on the other side of the stationary cross hair, *continuing* the count of the number of passages of the zero mark. Make another group of not less than ten settings, and use the mean of these in the same way as before in calculating the distance between successive division marks on the scale. Suppose this proves to be 3.7326. The fraction representing the position of the stationary cross hair between the two division marks is then $\frac{2.3012}{3.7326} = .61652$. This is a somewhat tedious operation, but it is advised because in this way the positions of the starting-point and finishing-point on the scale are thereby established with the degree of precision due to ten settings rather than one. The observation may, of course, be further refined by increasing the number of settings.

The starting-point having been thus established, the fringe count may be started. Observe and record each passage of a division mark under the stationary cross hair. Count a rather large number of fringes (say between a thousand and eleven hundred), then establish the finishing-point by a series of settings exactly similar to those made for the starting-point. The number of scale divisions passed over by mirror M_2 thus becomes known. The actual travel of M_2 is given by multiplying the known wave length of the monochromatic light by half of the number of fringes counted. Why? Hence the error in the part of the scale under observation may be calculated. Observation should be made in this way for a region at each end of the scale and one near the middle, and the percentage error of the scale at each point should be recorded, together with a statement as to whether the scale reads high or low.

B. MEASUREMENT OF REFRACTIVE INDEX OF A GLASS PLATE BY THE INTERFEROMETER

Apparatus. The interferometer to be used in this experiment should be equipped with frames to accommodate two extra glass plates, one in front of each of the mirrors M_1 and M_2. These frames should be so arranged that they can be rotated about a vertical axis. The one in front of mirror M_1 should be equipped with a slow-motion screw.

Two matched plates, optically plane parallel, of the specimen of glass whose index is required will be necessary. These should be of such dimensions that they fit the frames provided, and so mounted as to cover half — the same half, either upper or lower — of the field of view of the interferometer. We shall henceforth refer to these plates as *half-plates.*

The same auxiliary equipment will be needed for this as for the preceding experiment, with the exception of the tenth-millimeter scale and the microscope. In place of these there will be needed a small mirror and a reading-telescope and scale, to measure the angle through which one of the half-plates is turned. While both half-plates are to be turned through the same angle, it will be found most convenient to attach the mirror to that frame which is not equipped with the slow-motion screw.

Adjustments. Before attempting to take any readings, the student should become familiar with the process of adjusting the interferometer for fringes in white light. The search for these, particularly for the beginner, is liable to be somewhat tedious. Careful attention to the following instructions, coupled with the requisite amount of practice, will, however, rapidly decrease the time necessary for making the adjustment.

Adjusting for fringes in white light. It is better for the preliminary adjustment for fringes in white light not to have the half-plates in the instrument. Adjust so that the monochromatic fringes are arcs of circles. Move the carriage by intervals of a quarter-turn or less of the wheel *L.* When the region in which white-light fringes should be sought has passed, the curvature will have changed sign; that is, if the fringes were previously convex toward the right, they will now be convex toward the left. At the position which the central white-light fringe is to occupy, the corresponding monochromatic fringe will be perfectly straight, though it will be impossible in advance to determine this point exactly. Having thus located within narrow limits the position of mirror M_2 for white-light fringes (the position of zero difference of path), place a source of white light, such as a frosted electric-light globe, in such a position that it illuminates most of the field while a small part of the field is still occupied by the monochromatic fringes. Now move the mirror M_2 very slowly (if necessary, by the worm wheel S, which may be made to engage the handwheel L) through the region whose limits have already been determined. Since there are, as a rule, not more than a dozen of these fringes in white light, they often flash across the field too rapidly to be seen. Hence in the search for them it is wise to retain monochromatic fringes in part of the field as an index of the speed with which the mirror is being moved.

When the fringes in white light appear, they will exhibit the features described on page 114. In order that the central fringe may be entirely free of color, it is necessary that the plates H and C be cut from the same optically plane plate of glass and that they be strictly parallel to each other. When this condition exists, the central fringe will be achromatic, and the colors in the others will be symmetrically arranged with respect to the central fringe. For this experiment the instrument should be so adjusted that the fringes are vertical and sufficiently coarse so that each individual fringe is clearly visible.

It is not possible, under the conditions described here, to secure circular fringes in white light.

The small mirror having been previously attached to the appropriate frame, set the telescope and scale in place and make the necessary adjustments (see page 216).

Setting the half-plates perpendicular to the incident light. Place the half-plates carefully in their respective frames, the interferometer being adjusted for white-light fringes. It is not probable that the fringes will remain visible in that portion of the field covered by the half-plates (consider this to be the lower half). They may be rendered visible, however, by setting the half-plates parallel to M_1 and M_2, respectively, by simple inspection, and then slowly rotating one and the other on each side of this position until the fringes appear. When this occurs, it may be noted that if either of the half-plates is turned through a certain point, the direction of motion of the fringes undergoes a reversal. This marks the normal position of the plates; for when the plate is at right angles to the beam, its optical thickness is a minimum, and rotation either way will cause an increase in optical thickness. In this way set each of the plates in turn in its normal position. One of the half-plates, the one whose rotation is to be measured by the telescope and scale, must be set at the normal position with considerable accuracy. This is rendered somewhat difficult by the fact that in the region of the normal position the plate can be turned through a considerable angle before the fringes move appreciably. Therefore turn the plate in one direction until two or three fringes have passed (referring the positions of the fringes in the lower half of the field to the positions of the corresponding fringes in the upper half), and take the reading of the scale as seen in the telescope. Turn the plate in the other direction through the normal position until the same number of fringes have passed, and again take the reading. If the plate is now so placed that the scale-reading is halfway between these two, the precision of the normal setting will be quite sufficient. The white-light fringes should now be continuous, no "break" existing between the upper and lower half of the field. In case such a break does exist, it will be due to the fact that the plates are not parallel to each other vertically. "Rock" either plate in its frame about a horizontal axis until the break disappears. Note carefully the position and appearance of the central black fringe.

Measurements. Place the telescope and scale in such a position with reference to the instrument that the reading will be near one end of the scale. Bring into view the monochromatic source, at the same time retaining the white light so that the fringes from both sources are visible at the same time. Turn the half-plate which bears the mirror through such an angle that the other end of the scale appears in the telescope. The angle through which the plate is thus turned should be, if possible, more than 15 degrees. This motion of the plate will, of course, cause the white-light fringes to disappear from the lower half of the field, but the monochromatic fringes should still be visible. It is advisable to give the half-plate a position which will cause these monochromatic fringes to be continuous (that is, so that there is no break between the upper and lower halves of the field). The necessity for estimating fractions of a fringe is thereby avoided.

By means of the slow-motion screw, turn the half-plate which has heretofore remained in the normal position, at the same time counting the monochromatic fringes in the lower half of the field as they pass the central fringe of the upper half. When this half-plate has been turned through the same angle as was its mate, white-light fringes will reappear in the lower half of the field. The rotation should be continued until the upper and lower halves match in white light as completely as they did when the plates were in the normal position.

In this way the number of fringes $2N$ which pass as the half-plate is turned through a measured angle i becomes known. The thickness t of the half-plates may be measured with micrometer calipers. The value of the index of refraction μ of the plate is then given by equation (8).

CHAPTER X

THE PARALLEL–PLATE INTERFEROMETER

Interference in multiple beams. In Chapter IV we discussed the properties of the system of fringes produced by interference between the light from two slits. These fringes we observed to consist of an intensity distribution of the type pictured in Fig. 29. In Chapter V we observed the effect of multiplying the number of slits, the resulting device being termed a diffraction grating. The chief characteristic of this pattern was the narrowing of the maxima without affecting their separation, and hence the production of a pattern of the general nature of those shown in Fig. 33.

The pattern produced by the interferometer is identical with that due to the double slit except, as has been noted, that the rapid falling-off of intensity that is characteristic of the double-slit pattern is significantly absent. It has been found possible to produce a modification of the interferometer pattern that is quite analogous to the modification of the double-slit pattern when the number of slits is increased; that is, to cause the maxima to become narrower without affecting their distance apart, and hence so to modify the fringes that they present the appearance of a series of sharp, bright lines separated by relatively wide dark spaces.

The parallel-plate interferometer. Several devices exist that effect this modification, but they are all based upon the same principle, that of multiple reflection. The one that has come into the most common use is the interferometer usually named after Fabry and Perot.* This interferometer consists of two plane parallel plates, one stationary, the other movable in a direction perpendicular to its face. The adjacent surfaces are half-silvered. The course of a ray of light through these plates is shown in Fig. 65. After passing through plate *A*, part of it is reflected at the silvered surface of *B*, reflected again at that of *A*, and so on, until the diminution of its intensity by repeated reflections is such that the effect of added reflections may be neglected. At each partial reflection there is also a partial transmission, and hence every ray *S* gives rise to a whole series *S'*, between successive rays of which there exists a path difference equal to twice the separation of the mirrors (for perpendicular incidence).

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* It has sprung into common use as an adjunct of the grating in high-precision comparison of wave-lengths. It seems to be better adapted to this purpose than the Michelson interferometer, though to most other fields in which the latter instrument has made significant contributions to precise measurement, it is much less adaptable.

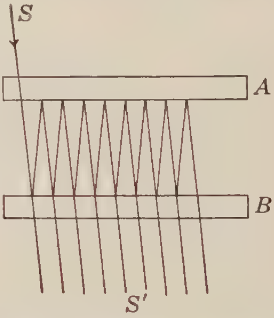

Fig. 65. Multiple reflection at two parallel surfaces

But these transmitted rays are not all equally intense, the losses in the successive reflections causing a regular decrease of intensity. It is a convenient assumption, and one which experiment justifies, that at each reflection the oscillations composing the light disturbance suffer a certain fractional loss σ in energy, and that hence the ratio of the intensities of the successive transmitted rays is a constant γ, which depends for its value upon the reflecting power σ (see equation (17)), but which must, of course, in every case possess a value less than unity.

Nature of the fringes. The fringes produced in this type of interferometer are, as has been stated, analogous to "orders" in grating spectra. They are not identical, however. Aside from the marked difference in dispersion, two factors predominate in effecting important differences.

One is the absence of the single-slit pattern, which causes the rapid falling off in the intensity of the successive orders of grating spectra. It is consequently possible, as in the Michelson interferometer, to observe many thousands of fringes when monochromatic light is used, instead of the very small number of fringes or orders which are visible in the grating spectrum.

The other is the fact that, whereas in the case of the grating all interfering beams possess the same intensity, in the Fabry-Perot interferometer the opposite is true. Consequently the intensity distribution of the interference pattern is notably different in the latter case from in the former.

The intensity function. Referring to Fig. 65, if we number the transmitted rays 1, 2, 3, · · ·, k · · · from left to right, the equation that describes the displacement s in the vibrations that constitute the kth ray is

$$s = c\gamma^{k-1}\sin 2\pi\left[\frac{t}{T} - \frac{\delta_k}{\lambda}\right] \tag{1}$$

(compare equation (1), Chapter III).

But the path difference δ_k between the 1st and the kth ray is, for perpendicular incidence,

$$\delta_k = 2(k-1)d, \tag{2}$$

where d is the distance between the half-silvered faces of the plates.

Hence
$$s = c\gamma^{k-1}\sin 2\pi\left[\frac{t}{T} - (k-1)\frac{2d}{\lambda}\right]$$

for one transmitted ray, and, for all,

$$S = \sum_{k=1}^{k=n} c\gamma^{k-1}\sin 2\pi\left[\frac{t}{T} - (k-1)\frac{2d}{\lambda}\right]. \tag{3}$$

Disregarding c for the time, the limit of this summation may be represented by a trigonometric series of the form

where
$$\left.\begin{aligned} &\underset{n\to\infty}{L}\sum_{m=0}^{m=n}\gamma^m \sin(\alpha - m\beta), \\ &m = k-1, \\ &\alpha = \frac{2\pi t}{T}, \\ &\beta = \frac{4\pi d}{\lambda}. \end{aligned}\right\} \tag{4}$$

This infinite series has the sum *

$$\Sigma = \frac{\sin\alpha - \gamma \sin(\alpha-\beta)}{1 - 2\gamma\cos\beta + \gamma^2}, \tag{5}$$

which may be expressed

$$A\sin\alpha + B\cos\alpha, \tag{6}$$

$$\left.\begin{aligned} &\text{where} & A &= \frac{1-\gamma\cos\beta}{1-2\gamma\cos\beta+\gamma^2}\\ &\text{and} & B &= \frac{\gamma\sin\beta}{1-2\gamma\cos\beta+\gamma^2}. \end{aligned}\right\} \tag{7}†$$

But (6) may be expressed in the form

$$D\sin(\alpha+\phi), \tag{8}†$$

$$\left.\begin{aligned} &\text{where} & D &= \sqrt{A^2+B^2}\\ &\text{and} & \tan\phi &= \frac{B}{A}. \end{aligned}\right\} \tag{9}$$

Hence the final expression for S becomes

$$S = Dc\sin(\alpha+\phi). \tag{10}$$

The intensity in the resulting interference pattern, being given by the square of the amplitude, is

$$I = c^2D^2 = c^2(A^2+B^2). \tag{11}$$

But $$A^2+B^2 = \frac{1}{(1-\gamma)^2 + 4\gamma\sin^2\frac{\beta}{2}}. \tag{12}†$$

$$\left.\begin{aligned} &\text{Therefore,} & I &= \frac{I_0}{1+C\sin^2\frac{\beta}{2}},\\ &\text{where} & I_0 &= \frac{c^2}{(1-\gamma)^2},\\ & & C &= \frac{4\gamma}{(1-\gamma)^2},\\ &\text{and} & \frac{\beta}{2} &= \frac{2\pi d}{\lambda}. \end{aligned}\right\} \tag{13}$$

(See the last of equations (4))

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* Todhunter, Trigonometry (edition of 1890), p. 246.

† The student is expected to verify equations (7), (8), and (12).

Maxima and minima. The value of I in the first of equations (13) is a maximum when $\frac{2\pi d}{\lambda} = m\pi$, that is, when

$$d = \frac{m\lambda}{2}, \text{ where } m = 0, 1, 2, \cdots; \tag{14}$$

and is a minimum when $\frac{2\pi d}{\lambda} = \frac{(2m+1)\pi}{2}$, that is, when

$$d = \frac{(2m+1)\lambda}{4}, \text{ where } m = 0, 1, 2, \cdots. \tag{15}$$

The minima are therefore midway between the maxima, and successive maxima or minima are produced by changing the separation d of the plates by $\frac{\lambda}{2}$.

Width of maxima. In investigating the width of the principal maxima in the case of gratings, the measure of the width was taken as the distance between the zero minima on each side. It is impossible to apply a similar definition here, for an examination of the first of equations (13) will show that no zero values of intensity exist. We shall accordingly take, as a measure of the width of the maxima, the distance between a point of maximum intensity and the point where this intensity has fallen to half of its maximum value. This is sometimes rather ambiguously termed *half-width.* It corresponds to the distance x of Fig. 66. The substitution of $\frac{I}{I_0} = \frac{1}{2}$ in the first of equations (13) will show that this corresponds to a value

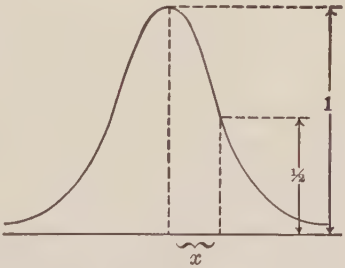

FIG. 66. "Half-width"

$$\frac{d}{\lambda} = \frac{1}{2\pi} \sin^{-1} \frac{1}{\sqrt{C}}. \tag{15'}$$

This is a function of C and hence of γ (see table, Fig. 66 A).

It will be shown (see succeeding paragraphs) that C is a large number, ordinarily about 100. Hence for the value of d corresponding to the half-width, we have a very small fraction of λ. But for a full fringe we have a change of $\frac{\lambda}{2}$ in d, and hence the width of the maximum is very small in comparison with the distance between fringes. That is, the fringes consist, as was stated in the early part of the chapter, of sharp, narrow maxima separated by broad minima. The student will show, after perusal of the following section, that the ratio of twice the half-width of a maximum to the distance between fringes is about .07 for the usual case.

γ	C	$\frac{\text{Min.}}{\text{Max.}}$	Half-Width
0.0	0.00	1.000	——
0.2	1.25	0.445	0.176 λ
0.4	4.44	0.184	0.079 λ
0.6	14.40	0.065	0.043 λ
0.7	31.00	0.031	0.028 λ
0.8	80.00	0.012	0.018 λ
0.9	360.00	0.003	0.008 λ
1.0	∞	0.000	0.000 λ

Fig. 66 *A*

Relation of intensity of minima to maxima. The intensity at all maxima is the same for strictly monochromatic light, and has the value I_0. The intensity at the minima may be seen, by examination of the first of equations (13), to bear to that of the maxima the ratio

$$\frac{\text{min.}}{\text{max.}} = \frac{1}{1+C}, \tag{16}$$

and hence depends upon the value of γ. The table of Fig. 66 *A* shows the value of this ratio for several values of γ.

Now the value of γ will quite plainly depend on the ratio σ between the intensities of a reflected and incident beam.

The relation is
$$\gamma = \sigma^2, \tag{17}$$

which the student is expected to verify. The value of σ will depend upon the thickness and quality of the silver films.

Experimentally it is found that films of such thickness that σ is about nine tenths are the most effective. This corresponds to a value of γ of about eight tenths. The corresponding value of the ratio of intensities of minima to maxima will be seen from the table of Fig. 66 *A* to be .012. The intensity of a minimum is hence about one one-hundredth that of a maximum. The intensity function (equations (13)) is plotted in Fig. 67, and shows the intensities of the minima and the values of the half-widths for several different values of γ.

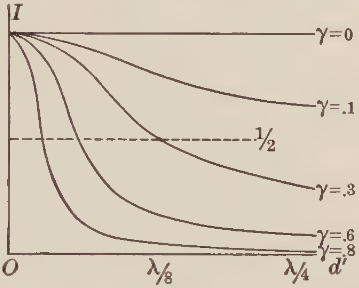

Fig. 67. Intensity distribution as function of transmissive power of Fabry-Perot interferometer plates

Form of fringes. The form of the fringes is governed by somewhat the same considerations as are discussed for the Michelson interferometer. For the use to which the Fabry-Perot instrument is usually put, however, the parallel position of the mirrors is preferable, and for this case the fringes are circular.

EXPERIMENT X

RATIO OF WAVE LENGTHS OF YELLOW LINES OF MERCURY

Apparatus. In addition to an interferometer of the type described in this chapter, there will be needed a mercury arc (with a filter to absorb all radiations except the yellow), a converging lens, and a hand counter. A filter of the type described on page 171 will be effective in case an appropriate gelatin filter is not available. Or a sodium source may be substituted for the mercury arc, and measurement made of the ratio of the wave-lengths of the D_1 and D_2 lines. This is, however, not quite as satisfactory, for the reason outlined on the following page.

Adjustments. It is even more important to *avoid touching the silvered surfaces* of this instrument than was the case with the Michelson interferometer, for the reason that more labor and greater skill are required in their preparation. For this reason, as well as to protect the instrument from dust, the interferometer should be inclosed, the adjustments and controls being outside.

The plates constituting the interferometer are mounted as described on page 130. Adjustments for securing parallelism of the plates are provided on the mounting of the plate nearest the observer. The other plate is movable in a direction perpendicular to its surface by means of a screw, as in the Michelson interferometer. In adjusting for fringes in monochromatic light, the two plates should be set as closely together as possible. With the source and lens so arranged that the field is uniformly illuminated, place a pointed object behind the plates, and bring its images into coincidence, both vertically and horizontally, by manipulating the plates with the adjusting-screws provided. When exact coincidence is secured, the fringes will spring into view.

With this type of interferometer, circular fringes are invariably used. Manipulate the adjusting-screws until the center of the system of circular fringes comes into view, continuing the adjustment until, as the eye is moved up and down as well as to the right and left, the circles remain unchanged in size. The ways in which the mounting of the movable mirror slides should be sufficiently true so that this adjustment remains unaffected as the mirror is moved back and forth.

If, now, the separation of the plates is varied, it will be found that the fringe system is less distinct for some locations of the movable mirror than for others. If a place of maximum distinctness is found and the fringes are observed, the features which distinguish them from the fringes of the Michelson interferometer may be verified. The bright portions of the field will not be as sharp as the remarks on pages 134 and 135 would lead one to suppose. The reason lies not alone in the lack of proper thickness and polish of the films. In the theory of the interferometer, as outlined in this chapter, a rigorously monochromatic source is postulated. Such a source does not exist, all spectral lines possessing finite width aside from that produced by diffraction, because they consist of an infinite number of wave-lengths. The effect of this lack of monochromatism is to broaden the bright portions of the fringe system by an amount that depends upon the distance between the mirrors. The lines of a sodium source, as ordinarily produced, are wider than those of a mercury arc and are therefore less adapted to take advantage of the properties of this type of instrument.

If the distance between the mirrors is now adjusted at some position other than maximum distinctness, and the fringes are carefully observed, it will be seen that there are two fringe systems visible, whose positions will be somewhat as shown in Fig. 68. As the separation between the mirrors is further altered, one of these fringe systems will be seen to move toward or away from the center faster than the other, so that the maxima of the two systems alternately coincide and lie between one another. The superposed position is the one of maximum distinctness. The same phenomenon occurs in the Michelson interferometer; but, owing to the fact that the light and dark portions are of equal width, the component fringes cannot be distinguished. In this instrument components are readily seen, and the positions of maximum or minimum distinctness are more easily located.

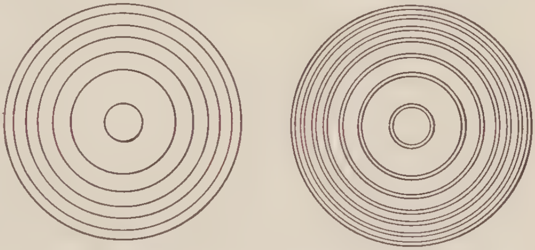

FIG. 68. Fabry-Perot interference pattern for two wave-lengths

With this type of interferometer it is not possible to produce fringes in white light.

Measurements. The distance between successive maxima or minima represents a change of path difference of one wave-length more of one radiation than of the other. If, therefore, the number of fringes n appearing or disappearing at the center is counted as the mirror is moved between successive maxima or minima, the fraction $\frac{n \pm 1}{n}$ represents the ratio of the two wave-lengths. It is a matter of indifference which sign is chosen, since any error introduced by wrong choice is well within the experimental error. Determine the value of n for at least five intervals. By using the wave-length ratio given by the mean of these determinations, the actual difference in wave-length may be calculated by assuming the value of one of the radiations.

CHAPTER XI

POLARIZED LIGHT

Polarization by reflection. The phenomena of diffraction and interference that have been treated in Chapters III–VIII are consequences of the undulatory nature of light, and themselves constitute the best evidence that light is a species of wave motion. Unfortunately, however, they fail to furnish evidence as to whether this wave motion is transverse or longitudinal. In fact, so far as the fundamental phenomena of reflection, refraction, diffraction, and interference are concerned, it would be difficult to distinguish sound, a purely longitudinal wave, from light, except for the marked difference in wave-lengths and the nature of the media which act as carriers.

There is, however, a class of phenomena, known as the phenomena of polarization, that differentiate light completely from sound, and show that light waves are not longitudinal, as are sound waves, but are transverse waves, which may be compared to those that elastic solids are able to propagate by virtue of their rigidity. These phenomena are so far removed from ordinary observation that they will be here presented in connection with a series of qualitative experiments.* The facts presented in the first experiment were discovered in 1810 by the French physicist Malus.

Experiment I. Set the plane-glass reflector *m* of the so-called Nörrenberg polariscope of Fig. 69 so that its plane makes an angle of about 33 degrees with the vertical. Adjust the position of a horizontal slit *s* (about 5 mm. wide) and a gas flame or frosted electric lamp *f* so that when you remove the black-glass mirror *m′*, and look vertically down upon the middle of *m*, you see a portion of the light. In this experiment the mirror *n* may be covered with a piece of black paper. Place *m′* in position, and turn it so that it is exactly parallel to *m*; that is, so that its plane also makes an angle of 33 degrees with the vertical. Place the eye at *E* in such a position that when you look at the middle of *m* you see the twice-reflected image of the light. Then rotate *m′* in its frame about a vertical axis, and observe the image of the slit as you do so. When you have turned *m′* through 90 degrees (that is, into the position shown in Fig. 69, 2), the image of the slit will have almost completely disappeared. If both *m* and *m′* have been recently polished and are perfectly clean, it may be possible to cause the image to disappear completely. Otherwise the surface impurities are sufficient to cause some light always to be reflected.

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* The directions for these experiments, as well as the accompanying explanatory remarks, have been taken, with some alterations and additions, directly from Chapter XXVII of "Electricity, Sound, and Light," by Millikan and Mills, to whom the author makes grateful acknowledgment of permission to use this material.

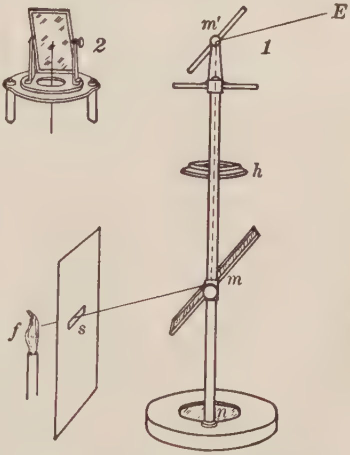

Fig. 69. Nörrenberg polariscope

Light reflected from the mirror *m* under these circumstances is said to be *plane-polarized*, and *m* is said to act as a *polarizer*. Since the mirror *m′* is used to investigate the properties of the plane-polarized light, it is said to act as an *analyzer*. A polarizer and an analyzer may be identical in construction, but are differently named by reason of the difference in their function.

The experiment suggests that light waves are not longitudinal, for if the particles of the medium that transmits the light from *m* to *m′* vibrated in the direction of propagation of the light, conditions of symmetry would demand that the wave be reflected in precisely the same way after *m′* had been rotated through 90 degrees as before. But if light consists of waves in which the direction of vibration of the particles of the medium is always transverse to the direction of propagation of the waves, and if in a very short interval of time the vibrating particles which give rise to light waves change their direction of vibration many times, then the above phenomena can be understood. For suppose that in Fig. 70, *ab*, *cd*, *ef*, *gh*, etc. represent successive directions of vibration of the particles of the medium across the path of the ray *sm*. All these vibrations can be resolved into two component vibrations: the one *perpendicular* to the plane of incidence* *smm′*, and represented by the dots in the line *sm*; the other *in* the plane of incidence, and represented by the straight lines drawn across the path of the ray *sm*.

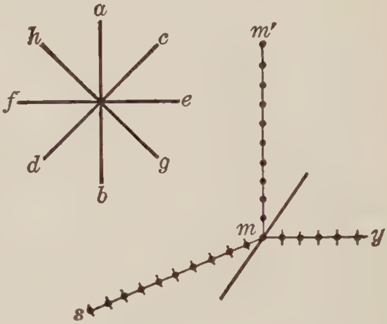

FIG. 70. Polarization by reflection

Now when the ray *sm* strikes the glass plate, part of it will be reflected (*mm′*) and part refracted (*my*). Certain experiments, whose nature cannot be considered here, indicate that reflection, instead of occurring *at* the surface separating two media, occurs after the light has penetrated the second medium to a small depth. Consequently it is *after* refraction that reflection really occurs. If we now consider light to be incident from such a direction that there is an angle of 90 degrees between the reflected and refracted rays, a little thought will make it evident that the reflected ray can consist *only* of vibrations perpendicular to the plane of incidence.

For of the two component vibrations forming the refracted ray, that which is in the plane of incidence is in the direction of the reflected beam, and hence, if it *could* constitute a part of the reflected beam, would appear as a *longitudinal* vibration. Our assumption of the absence of a longitudinal component in the vibrations constituting light prohibits this possibility; and the experimental fact that under proper conditions the once-reflected ray can be completely extinguished by a second reflection is a justification for this assumption. Hence, *at the angle of incidence for which the reflected and refracted rays are perpendicular to each other, the reflected ray consists only of vibrations that are perpendicular to the plane of incidence.* The light is said to have been plane-polarized by reflection at the *polarizing angle.* That this is always the angle for which the reflected and refracted rays are at 90 degrees was discovered in 1815 by Sir David Brewster, and is known as Brewster's law. It may be easily shown that another form of statement of the same law is the following: *the angle of complete polarization is the angle the tangent of which is the index of refraction of the reflecting substance.* This is the form in which Brewster announced his law. The student will show that the two forms of statement are equivalent.

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* The plane of incidence is defined in general as any plane parallel to that containing the incident ray and the normal to the reflecting or refracting surface.

The reason why we originally set the mirror *m* to make an angle of 33 degrees with the vertical was that the index of refraction of crown glass is about 1.55, and the angle the tangent of which is 1.55 is 57 degrees. In order that the angle of incidence might be 57 degrees it was necessary to make the angle between the plane of the mirror and the vertical 33 degrees.

It will now be seen why *m′* gave no reflected light when rotated from position 1 to position 2. For in this latter position *m′* bore precisely the same relation to the vibration of the ray *mm′* as did the plate *m* to the component of *sm* which was vibrating in the plane of incidence *smm′*.

Experiment II. Now set *m* and *m′* again so that there is no light from *f* reflected in *m′*, and then rotate *m′* about a *horizontal* axis, observing the middle of *m′* all the while. You will find that there is always some of the ray *mm′* reflected from *m′*, except when *m′* is set exactly at the polarizing angle. The amount of light thus reflected will be found to increase rapidly as the position of the mirror departs in either direction from the polarizing angle.

Experiment III. Replace the glass mirror m' by a pile of about fifteen thin glass plates set at the polarizing angle (see Fig. 71), and then observe not only, as above, the reflected ray e, but also the ray e' transmitted by the plates as the pile is turned about a vertical axis. You will find that when the pile of plates is in position 2 (Fig. 69) (that is, in a position such that the reflected ray disappears), the transmitted ray is of maximum brightness; and that when the plates are rotated into position 1 (that is, such a position that the reflected ray is of maximum brightness), the transmitted ray has almost entirely disappeared.

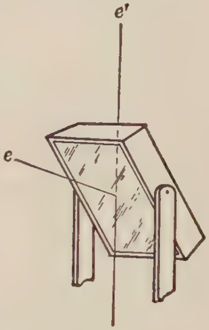

Fig. 71. Glass-plate pile as analyzer

In explanation of these effects consider that an incident beam *sm* of ordinary light (Fig. 72) is resolved into two components, one vibrating *in* and one *normal to* the plane of incidence. Let the intensity of each of these components be represented by 50. At the polarizing angle none of the vibrations which lie in the plane of incidence are reflected, while photometric measurements show that about 16 per cent of the light which consists of vibrations perpendicular to the plane of incidence is reflected; that is, 8 per cent of the incident beam is reflected at the polarizing angle. Hence, after the first refraction the transmitted light consists of fifty parts vibrating in the plane of incidence and forty-two parts vibrating in the plane perpendicular to the plane of incidence. After the second refraction these numbers have become 50 and 35.3; after the third refraction, 50 and 29.7; after the fourth, 50 and 25; and so on. After passage through twelve or thirteen plates the transmitted light has become nearly plane-polarized by the process, the plane of its vibrations being

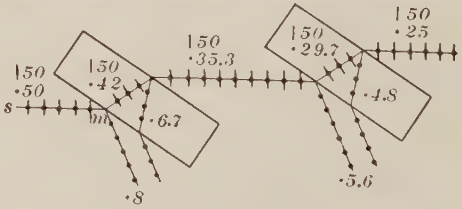

Fig. 72. Polarization by transmission

obviously at right angles to the plane of vibration of the reflected light. A pile of plates of this sort furnishes a very inexpensive means of obtaining plane-polarized light, but it suffers from the disadvantage that the polarization is not quite complete. If no light whatever were absorbed or scattered by the glass, the transmitted ray would become more and more nearly plane-polarized the larger the number of plates, but in practice there is found to be no advantage in increasing the number beyond thirteen or fourteen.

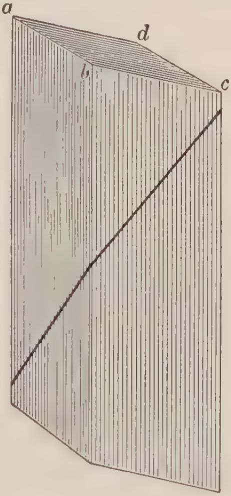

FIG. 73. Nicol prism

Experiment IV. Replace the upper mirror by a Nicol prism (Fig. 73), the construction of which will be considered later; and, looking through it at the image of *f* reflected in *m*, rotate it about a vertical axis. You will find that the ray *mm'* is cut off completely by the Nicol when the latter is in a certain position, but that the light is transmitted with maximum brightness when the Nicol has been rotated through an angle of 90 degrees from this position. From a knowledge of the plane of vibration of the ray *mm'* decide what must be the plane of vibration of the ray with respect to the face *abcd* of the Nicol in order that it may be wholly transmitted by the latter, and mark the direction of this transmitting plane by an arrow drawn on the face of the mounting containing the Nicol. Henceforth you can use the Nicol as an analyzer of polarized light.*

Polarization by double refraction. The phenomena which will be presented in the following experiments were discovered in

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*According to the electromagnetic theory of light a polarized ether wave consists of an electric vibration (*displacement current*) in one plane accompanied by a magnetic vibration in a plane at right angles to it. The "light vibrations," as somewhat loosely spoken of in this chapter, are to be identified with the electric vibrations, and the direction of these vibrations is *perpendicular* to what is commonly termed the *plane of polarization*. To avoid ambiguity we shall continue to speak of the "plane of vibration" rather than the "plane of polarization," with the understanding that they are related as above.

1670 by the Danish physicist Erasmus Bartholinus, who was the first to notice the fact of double refraction in Iceland spar, and in 1690 by Christian Huygens, who was the first to notice the polarization of the doubly refracted beams produced by the Iceland spar, and to offer an explanation of double refraction from the standpoint of the wave theory of light.

Experiment V. Make a pinhole in a piece of black cardboard, and lay the cardboard on a piece of plane glass on the frame *h* (Fig. 69). Several inches beneath this (for example, on the plate *m* turned to a horizontal position) lay a piece of white paper and illuminate it well. Then lay a crystal of Iceland spar (Fig. 74) over the hole in the cardboard. Remove *m'* and look vertically down upon the crystal. You will see two pinholes instead of one. Rotate the crystal about a vertical axis. One image will remain stationary, while the other will rotate about it. It will require some care to ascertain which is the fixed and which the moving image.

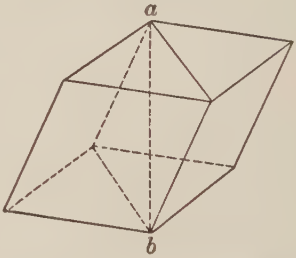

Fig. 74. Equilateral crystal

That the image which remains stationary is produced by light which has followed the usual laws of refraction is evident from the fact that it behaves in all respects as it would if viewed through a glass plate. The image which rotates, however, must be produced by light which has followed some extraordinary law of refraction; for although it has passed into the crystal in a direction normal to the bottom face, and out of it in a direction normal to the top face, it must have suffered bending inside the crystal, since it emerges at a point different from that at which the other ray emerges. We must conclude, then, that a ray of light which is incident upon the lower face of such a crystal of Iceland spar is split into two rays by the spar, and that these two rays travel in different directions through the crystal. The ray which follows the ordinary law of refraction is called the *ordinary* ray; the other, the *extraordinary* ray.

Experiment VI. To find the direction in which the extraordinary ray travels, rotate the crystal about a vertical axis and note that the extraordinary image lies approximately in the line connecting the ordinary image and the solid obtuse angle of the face which is being viewed, and, further, that the extraordinary image is always on that side of the ordinary image which is away from this solid obtuse angle.

It will be evident, then, from these experiments that if Fig. 75 represents a section of the crystal made by passing a plane normal to the top and bottom faces and parallel to the optic axis of the rhomb, the line *io* will represent the path of the ordinary ray through the rhomb, while the broken line *imne* will represent the extraordinary ray.

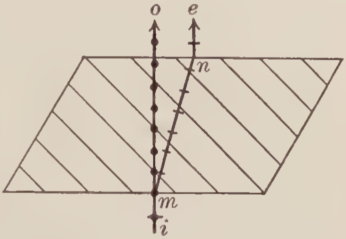

FIG. 75. Polarization by double refraction

Experiment VII. In order to determine whether the ordinary or extraordinary ray travels faster through the rhomb, observe again the two pinholes; or, better, observe at close range a dot on a piece of white paper upon which the crystal lies, and note which image, the ordinary or the extraordinary, appears to be nearer to the upper face.

The image nearer to the upper face is evidently associated with rays which have suffered the largest change of velocity upon emergence; that is, it corresponds to rays which travel the more slowly in the crystal. These will be found to be the ordinary rays. In some crystals, of which quartz is the most common example, the extraordinary ray travels the more slowly. To distinguish the two types the former is usually termed *negative*; the latter, *positive.*

Experiment VIII. View the pinhole through a crystal whose faces have been cut perpendicular to the optic axis. You will observe that there is now but one ray, and that this ray does not change position as the crystal is rotated; that is, that it behaves in the ordinary way.

It is thus seen that when light passes through this crystal in a certain direction, the phenomena of double refraction are not evident. This direction is the *optic axis* of the crystal.

The optic axis possesses a definite relation to the geometry of the crystal. When split along natural planes of cleavage, crystals of Iceland spar assume the form of rhombohedrons whose dimensions will vary but whose angles are always the same. In each rhombohedron there are two corners where three obtuse angles come together. A line drawn through one of these corners, equally inclined to the three faces, *or any line parallel to it*, may be regarded as the optic axis of the crystal. In Fig. 74, which represents an equilateral rhombohedron, the lines *ab* through opposite solid obtuse angles are coincident.

Experiment IX. Place the Iceland spar again over the pinhole, in the manner indicated in Experiment V, and view the two images through the Nicol prism as the latter is rotated about a vertical axis. You will find that both the ordinary and extraordinary images consist of plane-polarized light, but that the planes of vibration of the waves which produced the two images are at right angles to one another.

Hence we may conclude that the Iceland spar has in some way separated the incident light into two sets of vibrations, one of which consists of those components of the initial vibrations which were parallel to a particular plane in the crystal, while the other consists of those components of the initial vibrations which were perpendicular to this plane.

Experiment X. With the aid of the Nicol, whose transmitting plane you determined in Experiment IV, find whether the ordinary or the extraordinary ray consists of vibrations which are parallel to the plane that includes the optic axis of the crystal and the line of sight.

You will find that it is the extraordinary ray whose vibrations are in this plane, while those of the ordinary ray are perpendicular to this plane (see Fig. 75).

Theory of double refraction. The elementary theory of double refraction is as follows: For the sake of simplicity we shall confine attention to the wave form in a single plane in the crystal; namely, the plane which is perpendicular to the upper and lower faces of the crystal and includes the optic axis. This is called a *principal plane.* It is the plane of the paper in Figs. 76, 77. As we have already seen, any incident beam of light which passes normally into the crystal through the hole in the cardboard (see Figs. 76, 77) may be thought of as consisting of equal vibrations in two planes, one perpendicular to the paper and the other parallel to it. Let us consider these two vibrations as separated, so that we may treat of one in Fig. 76 and the other in Fig. 77. Let us suppose, further, that any vibrations which are parallel to the optic axis pass through the crystal with greater speed than do vibrations which are perpendicular to this direction; that is, we have a negative crystal.

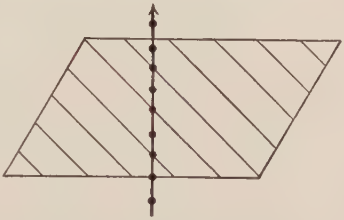

FIG. 76. Course of ordinary ray through a crystal

The component vibrations perpendicular to the principal plane, being also perpendicular to the optic axis (see Fig. 76), will travel with equal speed, whatever the direction of the corresponding rays. The propagation of these rays will therefore be in no way different from the propagation of light in isotropic media. The light will follow the usual law of refraction and for zero angle of incidence will undergo no change in direction. This is the condition that defines the ordinary ray.

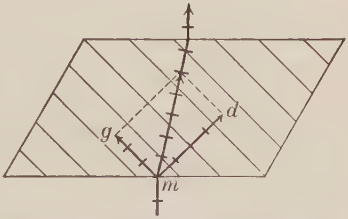

FIG. 77. Course of extraordinary ray through a crystal

On the other hand, the component vibrations parallel to the principal plane are not necessarily at the same time parallel to the optic axis (see Fig. 77). They may, however, be resolved into two components, one parallel and the other perpendicular to the optic axis. Since the parallel component *md* travels faster than the perpendicular one *mg*, the resultant beam will suffer a change in direction upon entering the crystal, even though the angle of incidence be zero. This is the condition that defines the extraordinary ray.

The fact that the velocity of a ray through a crystal depends upon the direction of its vibrations with reference to the optic axis involves as a consequence the fact that the index of refraction of the crystal is different in different directions. For a negative crystal this index has a minimum value for rays whose vibrations are parallel to the optic axis, and a maximum for those whose vibrations are perpendicular to that axis. The index assumes all possible values between this maximum and minimum, depending upon the direction in which light traverses the crystal. The maximum and minimum values themselves are usually termed the *principal indices of refraction.*

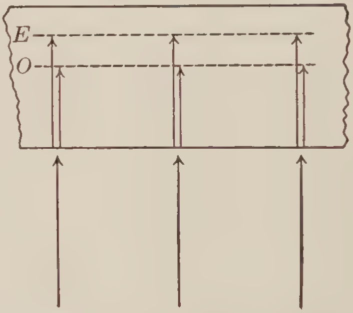

Fig. 78. Action of half-wave plate

If a crystal is so cut that the optic axis is parallel to the surface and in the plane of the paper (Fig. 78), then there will be no lateral displacement of the image, for both the ordinary and extraordinary rays travel in the same direction. The two rays will still traverse the crystal with different velocities, however, with the result that the extraordinary wave front will be ahead of the ordinary wave front by an amount depending upon the thickness of the crystal. If the crystal is of such a thickness that one wave front is a quarter of a wave-length ahead of the other, it is called a quarter-wave plate; if such that one is half a wave-length ahead of the other, a half-wave plate; etc.

In actual manufacture of these plates it is not necessary to cut the crystals so that their faces are parallel to the optic axis, for they are so thin that the separation of the ordinary and extraordinary rays is imperceptible. It is much simpler and equally effective to split them along their natural planes of cleavage.

Construction of the Nicol prism. In order that the light which is transmitted by a crystal of Iceland spar may consist of vibrations in one plane only, it is necessary to dispose in some way of either the ordinary or extraordinary beam so as to prevent it from emerging from the crystal. This was first accomplished in 1828 by the Scotch physicist Nicol, in the following way: If the beam *bc* (Fig. 79) is made to enter the face of the crystal at a certain oblique angle, the ordinary ray, being refracted more than the extraordinary (see Experiment VII), will travel in the crystal in the direction *co*, for example, while the extraordinary ray will take the direction *ce*. Nicol cut the crystal into two parts along the plane *aa*, and then cemented the parts together again with Canada balsam. This balsam has an index of refraction which is smaller than that of Iceland spar for the ordinary ray, but larger than that of the spar for the extraordinary ray; hence it was possible, by using a long crystal like that shown in the figure, to choose the plane *aa* so that the ordinary ray would be totally reflected, and absorbed in the blackened walls of the crystal, while the extraordinary ray would pass through.

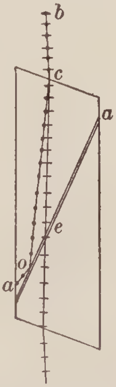

FIG. 79. Action of Nicol prism

Construction of the Wollaston prism. One of a rather large number of fabricated double-image prisms is the Wollaston prism. This consists, like the Nicol, of two crystal prisms cemented together. In the Wollaston, however, the optic axis of the first prism is parallel to the surface. Consequently, although the ordinary and extraordinary rays (in the case of normally incident light) have the same direction, their velocities are different. In the second prism the optic axis, while still perpendicular to the incident ray, is also perpendicular to that of the first prism (see Fig. 80). On entering this part the ordinary ray becomes the extraordinary and vice versa. The two rays therefore interchange velocities; that is, one ray passes from a medium of smaller refractive index to a medium of larger, and is hence refracted *toward* the normal, while the other, for the contrary reason, is refracted *away from* the normal. This phenomenon, combined with a second refraction upon emergence, causes the two rays to diverge, not only upon entering the second portion of the crystal, but still more so upon leaving it.

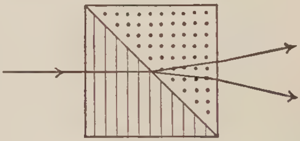

FIG. 80. Action of Wollaston prism

Experiment XI. Observe the double image of a pinhole through a Wollaston prism. It will be observed that the distance between the images depends upon the distance between the prism and the object, and that hence the emergent rays diverge as shown in Fig. 80. Moreover, as the prism is rotated, *both* images will be seen to revolve about the mid-point of the line joining them. Compare this with the similar phenomenon in Iceland spar. Observation of these images through a Nicol will show that light from one is polarized in a plane perpendicular to the line joining them, and from the other in a plane parallel to the line joining them. This is evidence that the optic axis of the emergent half of the prism must be either parallel or perpendicular to the line joining the two images.

Passage of polarized light through thin crystals. This class of phenomena may best be introduced by experiment.

Experiment XII. Arrange the polarizing apparatus as shown in Fig. 69, save that the Nicol prism replaces the mirror m' and that the source of illumination is a sodium flame. Rotate the Nicol until the flame is completely extinguished. Place on the slide-holder h a half-wave plate; you will find that as a rule this will cause the light to reappear. Rotate the plate about a vertical axis and note that in one revolution there are four positions, 90 degrees apart, at which there is extinction. These are the positions in which the plane of vibration of the light is either parallel or perpendicular to the plane containing the optic axis. Place the plate in a position 45 degrees from one of these positions of extinction. The field will appear bright when regarded through the Nicol. Rotate the Nicol until extinction is again secured. It will be found that the Nicol has been rotated through 90 degrees. Experiment will show that in general a rotation of the plate through a given angle, starting from one of the positions of extinction, will rotate the plane of polarization through twice that angle.

We may explain this phenomenon as follows: In Fig. 81 let *a* represent the plane of vibration of the light which is incident upon the half-wave plate. Let *b* represent the two vibrations into which the incident vibration is decomposed as it enters the crystal, the one, *e*, in the plane of the optic axis and the other, *o*, perpendicular to this plane. The fact that upon emergence one of the wave fronts will have fallen behind the other by half a wave-length, and that hence its phase with respect to the other will have changed by 180 degrees, is indicated at *c* by changing the position of the arrowhead, on the line which represents this vibration, from one end to the other. These two vibrations, upon emergence into the air, will recombine into a single vibration, which is represented at *d*. The light which emerges from the crystal will therefore be still plane-polarized, but the direction of its vibration will be (for the 45-degree position of the plate) at right angles to the direction of vibration of the beam when it entered the crystal. Since the Nicol was originally set to cut out the vibrations in the direction of *a*, it will of course transmit with maximum intensity vibrations parallel to *d*. In order to extinguish this vibration it should be necessary to rotate the Nicol through 90 degrees, as we found in our experiment was the case.

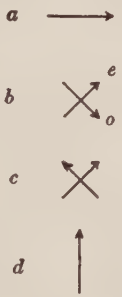

Fig. 81. Action of a half-wave plate set at 45 degrees to plane of polarization

If the plate had been half as thick as it was, the two components would have emerged from the crystal a quarter of a wave-length apart instead of half a wave-length. Under these circumstances the vibrations constituting the two emerging wave fronts would be out of phase with each other by 90 degrees. It is a theorem of elementary mechanics that the combination at right angles, of two simple harmonic vibrations having equal amplitudes and periods, and possessing a phase difference of 90 degrees, produces uniform circular motion. The situation is rather crudely represented in a graphical way in Fig. 82. A little consideration will show that the analyzing Nicol may be expected to transmit the same fraction of the intensity produced by this circular vibration, whatever its position.

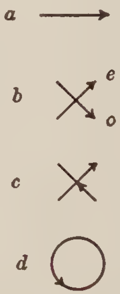

Fig. 82. Action of quarter-wave plate set at 45 degrees to plane of polarization

Experiment XIII. Replace the half-wave plate by a quarter-wave plate. In the same way as before, set it at 45 degrees from a position of extinction; then rotate the Nicol and note that no change takes place in the intensity of the transmitted light.

The above experiment illustrates the production and identification of *circularly polarized* light. Since the rotation of the analyzer fails to produce any change in intensity for this case, it would appear at first thought that there is no distinction between circularly polarized and unpolarized light. If, however, a second quarter-wave plate is inserted in the path of the circularly polarized light, one component will undergo a second phase-change of 90 degrees, and the emergent beam will become again plane-polarized. No such phenomenon would be observed if a quarter-wave plate were placed in the path of an *unpolarized* beam.

Experiment XIV. Verify the statements of the foregoing paragraph. Rotate the second quarter-wave plate and observe that the plane of vibration of the emergent light is rotated by the same amount. Account for this fact by reversing the process of reasoning suggested by Fig. 82.

If we introduce into the path of a plane-polarized beam a plate of such thickness as to produce a retardation (of one component with respect to the other) of one wave-length, two wave-lengths, three wave-lengths, and so on, the light as it emerges from the crystals will vibrate in the same plane as that in which it entered. Hence it will be completely cut out by the Nicol when the latter is set in the position of extinction for the case in which no crystal is interposed.

If the plate is of such thickness as to cause a retardation which is not equal to an integral number of quarter-waves, the light which emerges from the crystal will be *elliptically polarized*; that is, the vibrations of the ether particles take place in the form of ellipses. For the same considerations of mechanics as were referred to above, show that when the phase difference is not 90 degrees, the resultant is in general an ellipse. A little consideration will show that elliptically polarized light could also be produced by a quarter-wave plate oriented at some other than a 45-degree angle. The major and minor axes of this elliptical vibration may be located by observing the orientation of the analyzing Nicol when the intensity is a maximum or a minimum. The subject of elliptically polarized light will be further discussed in Chapter XIII.

In some types of optical measurement wave plates whose thickness is variable are very convenient. Such devices are often used to compensate for a phase difference already introduced, thereby restoring a previously plane-polarized beam to its original condition. Because of their use in this way, these devices have become commonly known as *compensators.*

The most common form is the Babinet compensator. It consists of a pair of thin quartz wedges of the same angle, oppositely mounted, so that the combination forms a thin plane-parallel plate. Variable thickness of this plate is secured by so mounting one of the wedges that it may be moved by a micrometer screw in a direction perpendicular to its refracting edge. Both wedges are so cut that the outer face is parallel to the optic axis, but with the axes at right angles in the two wedges (compare the Wollaston prism). Compensation is obtained in narrow bands which, when viewed between crossed Nicols, present the appearance of interference fringes. (This device is discussed more in detail in Appendix II.)

Another form of compensator, the Soleil, consists of a pair of wedges cut alike and combined with a plate cut with the axis perpendicular to the axis in the wedges. In this device the whole field presents a uniform shade.

Experiment XV. Set the analyzer for extinction. Place in the path of the plane-polarized monochromatic beam a quartz or selenite wedge whose axis is parallel to its sides, setting it at 45 degrees with the plane of vibration of the beam. It will be crossed by a number of bands parallel to its refracting edge. Rotation of the analyzer through 90 degrees will cause the light and dark bands to be interchanged. The points midway between the light and dark bands remain unchanged in intensity throughout the process. If white light is substituted for monochromatic light, these bands become colored.

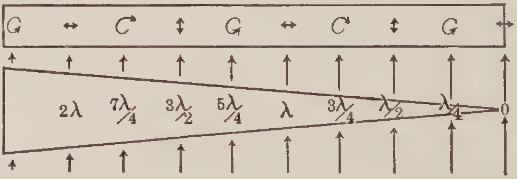

FIG. 83. Action of crystal wedge oriented at 45 degrees on plane-polarized light

Fig. 83 represents an attempt to show by a diagram the facts which account for the phenomena thus observed. The positions marked $\frac{\lambda}{2}$, $\frac{3\lambda}{2}$, etc. are those parts of the wedge which perform the functions of a half-wave plate. The positions marked $\frac{\lambda}{4}$, $\frac{3\lambda}{4}$, etc. perform the functions of a quarter-wave plate. Light traversing the portions 0, λ, 2 λ, etc. remains unaltered, for reasons described above. Turn the wedge so that its axis is first parallel and then perpendicular to the plane of vibration of the incident light, and observe it both when the Nicol is set for extinction and when it is set for transmission. The vibrations being for both cases either parallel or perpendicular to the plane containing the axis and the ray, the light is unaffected by passage through the crystal. A discussion of the color phenomena observed in white light may profitably be postponed until the student has become acquainted with the facts set forth in the following paragraph.

Experiment XVI: colors of thin crystals in polarized light. For this experiment either artificial white light or daylight may be used. If the former is employed, the source should have considerable extent and may profitably be a strongly illuminated sheet of ground glass. If daylight is used, set the polariscope in a window in somewhat the same position as in Fig. 84, the black paper having been removed from the mirror *n*. If *m* has the same inclination as in Experiment I, when the polariscope is so turned that the prolongation of the line *ma* meets the sky, the white light from the sky will strike the lower side of *m* at the polarizing angle, be reflected to the mirror *n*, and return with little loss, as a plane-polarized beam, to the Nicol *N*. Set *N* so that this beam is extinguished. Place a sheet of mica upon *h*, and turn it until it is 45 degrees from a position of extinction. When viewed through the Nicol it will be found to exhibit colors. Rotate the Nicol slowly and notice that a rotation of 45 degrees causes the colors to disappear, but that a rotation of 90 degrees causes colors which are complements of the first colors to appear. Further rotation through 90 degrees will cause the first colors to return, and so on. When sheets of mica of different thicknesses are used, different colors will be produced, and different portions of the same sheet will usually appear of different color; but a rotation of the Nicol through 90 degrees will always cause the colors to change to the complements of the original colors.

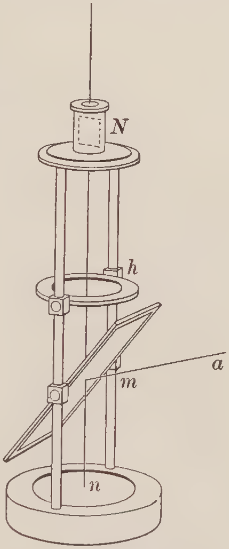

Fig. 84. Method of applying diffuse daylight in polariscope

In order to understand the cause of this phenomenon, suppose, for simplicity, that the mica is just thick enough to produce a retardation of the ordinary ray, relatively to the extraordinary, of half a wave-length of the longest red wave-length. Since the shortest violet waves have about half the wave-length of the longest red, this same plate will produce a retardation in the violet of one whole wave-length. The violet wave will therefore emerge from the crystal with both its components in the same phase, and these components will recombine into a plane vibration precisely like that which entered the crystal; namely, the vibration represented in Fig. 81, *a*. The red ray, however, will emerge from the crystal with one of its components half a wave-length behind the other, and these two components will recombine into a vibration at right angles to that of the entering ray; that is, after the manner of Fig. 81, *d*. If, then, the Nicol is in the position for extinction when no crystal is interposed, it will cut out all the violet in the incident white light and transmit all the red, so that if these red and violet waves alone fell upon the crystal, rotations of the Nicol through 90 degrees would cause red and violet to appear alternately. As a matter of fact, however, if the incident light is white, all the colors between the red and the violet will be present, and the vibrations of the transmitted light which correspond to them will be ellipses of some form. However, the wave-lengths which are close to the red (namely, those of the orange and yellow) will be largely transmitted along with the red and will have but small components to be transmitted with the violet; while the wave-lengths near the violet (namely, the blues and shorter greens) will be mainly transmitted with the violet. Hence the light which passes through the Nicol in the first position will be some shade of red, because it will lack the shorter wave-lengths; while when the Nicol is turned through 90 degrees, all the wave-lengths which were before cut out will be transmitted. The color will therefore be the complement of the first, — some shade of blue.

A crystal which is too thin to produce a retardation of half a wave-length in the shortest visible rays cannot show any marked color effects in polarized light, since no wave-length can be entirely cut off for any position of the Nicol. On the other hand, a crystal which is so thick as to produce retardation of very many wave-lengths of any one color will produce also a retardation of an exact number of wave-lengths for each of many other colors scattered throughout the spectrum. These colors will all be cut out by the Nicol, and the transmitted light will likewise consist of wave-lengths which are taken from all parts of the spectrum, and will therefore reproduce the effects of white light. Hence these *color phenomena in polarized light can be observed only with crystals which produce a small number of wave-lengths of retardation.* By scraping down crystals to proper thicknesses in different parts, color patterns of great beauty are often produced when the crystals so treated are viewed in polarized light. All the colors, of course, change to their complements upon rotation of the Nicol through 90 degrees.

Experiment XVII. Place a number of these designs in selenite, quartz, or mica upon the slide-holder, and observe the appearance of the complementary colors in different portions of the design as the Nicol is rotated.

Double refraction produced by strain. Double-refraction phenomena are sometimes apparent in isotropic media. One method of producing them is described in the succeeding experiment.

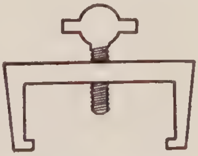

Fig. 85. A vise for producing double refraction in isotropic transparent bodies

Experiment XVIII. Place a piece of glass or xylonite in the frame provided (see Fig. 85) and subject it to a bending stress, at the same time observing it in the polariscope. The colors will be most marked when the line of the stress makes an angle of 45 degrees with the plane of vibration of the incident light. Observe, also, without using the clamp, a piece of unannealed glass; that is, a piece of glass which has been subjected during the process of manufacture to a sudden cooling. A symmetrical color pattern will be observed.

The production of double refraction by strain in transparent objects has been made useful in two ways. Professor E. G. Coker has investigated the distribution of stresses in bridge members by observing the colors produced in polarized light in xylonite models placed under stress. Observation in polarized light is also one of the standard tests of quality in optical glass. The production of double-refractive effects by strain is not completely understood, though it is commonly attributed to the effect of strain on the arrangement of the molecules of which the substance is composed.

Polarized-light effects in crystals cut perpendicular to axis. A group of phenomena entirely distinct from those described above are associated with the passage of plane-polarized light through crystals so cut that their faces are perpendicular to the optic axis. Some of the more simple of these are described in connection with the succeeding experiments.

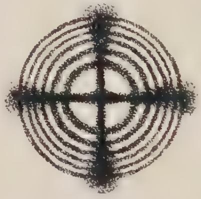

Fig. 86. Uniaxial crystal, cut perpendicularly to axis, viewed in convergent light, Nicol set for extinction

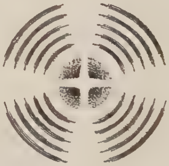

Fig. 87. Same as the preceding figure but with the Nicol set at 90 degrees to position of extinction

Experiment XIX. Observe in convergent polarized light a crystal of Iceland spar, about 1 mm. thick, the upper and lower faces of which are planes perpendicular to the optic axis. The lenses mounted on the polariscope may be used for this purpose, or the crystal may be placed very close to the Nicol in the arrangement of Fig. 84, so that the observer looks down through the crystal on a field of considerable width, from all parts of which polarized light is approaching the eye. If the Nicol was originally set for extinction, you will observe a dark center surrounded by a series of brilliantly colored rings upon which is superposed a black cross (see Fig. 86). Rotating the Nicol will cause the black cross to change to a white one, and all the colors to change to their complements (see Fig. 87). These figures are often termed *rings and brushes.*

These effects may be explained as follows: The central rays pass through the crystal in a direction parallel to the optic axis. Their vibrations therefore suffer no resolution into ordinary and extraordinary components and hence no change in the character of their vibration. They are cut out by the analyzing Nicol; hence the black center. The light, however, which converges upon the eye after passing through the outer edges of the crystal has traveled in directions slightly oblique to the axis and has therefore suffered decomposition into ordinary and extraordinary rays, which have undergone different retardations. A given retardation of one set of rays with respect to the other corresponds to a given color, as explained above. The effects are quite analogous to the phenomena observed in the quartz wedge, except in this respect: in the wedge, progressive retardations were caused by increasing thickness, whereas in the present case they are caused by increasing obliquity of light passing through a plate of uniform thickness. A given color must, of course, be symmetrically distributed about the axis of the converging beam, since the thickness traversed by successive cones of light is so distributed; hence the concentric rings of color. The black cross is superposed upon the rings for the same reason that the wedge became uniformly black when rotated to a certain position with the Nicol set for extinction; that is, because in two particular planes (namely, those for which the incident vibrations are respectively in and perpendicular to the plane containing the axis and the ray) even these oblique rays are not split up into components, but pass through, vibrating in their original direction, and are therefore cut out by the Nicol. Upon rotating the Nicol through 90 degrees all these previously extinguished rays are, of course, transmitted; hence the white cross.

Certain crystals possess two optic axes at an angle with each other, and are hence termed *biaxial*, in contradistinction to the *uniaxial* crystals which we have heretofore considered. These therefore possess three principal indices of refraction in place of the two which were characteristic of uniaxial crystals. Such crystals, when cut perpendicular to the bisector of the angle between the optic axes and observed in convergent polarized light, present a series of curves of equal retardation of the form of Fig. 88, which therefore represents the general appearance of the field. There are present two *brushes*, approximately hyperbolic for one position of the crystal.

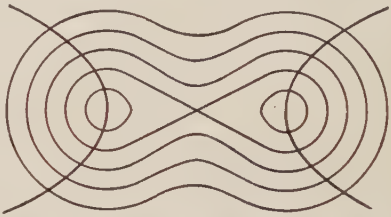

FIG. 88. Curves of equal retardation in biaxial crystal

The classification of crystals according to the phenomena which they exhibit when observed in polarized light, forms an important branch of geology. It is not always easy to secure crystals of sufficient size to exhibit their properties to the unaided eye. Special microscopes which possess, among other auxiliaries, polarizing and analyzing attachments to facilitate these studies in polarized light, are commonly known as *petrographic* microscopes, and are indispensable in this field of study.

Rotation of the plane of polarization. Under appropriate conditions the plane of polarization undergoes a rotation when polarized light passes through certain substances. The following experiments are intended to emphasize some of the outstanding features of this phenomenon.

Experiment XX. Arrange the polariscope as in Fig. 69, save that m' is replaced by a Nicol, and the source is a sodium flame. Place upon the slide-holder a crystal of quartz about five millimeters thick, the upper and lower faces of which are made by planes which are at right angles to the optic axis of the quartz. When the Nicol is set for extinction, the introduction of the quartz into the path of the beam will be found to cause the image of the flame to reappear. Rotate the Nicol and measure the amount of rotation required to cause the flame to disappear again. According to accepted results this rotation for sodium light should be 21°.7 per millimeter of thickness of the quartz. Replace the sodium flame by the ordinary violet flame of the burner and repeat, rotating this time until all trace of the violet color has disappeared from the image of the flame. The rotation will be found to be about double that found for sodium light. The rotation for white light filtered through red glass will be found less than that due to sodium light.

These experiments show, first, that plane-polarized light which passes through quartz in the direction of its optic axis remains plane-polarized after transmission; and, second, that the plane of vibration of the light has been rotated by the quartz, the amount of rotation being greater for the short wave-lengths than for the long. It has been found that this *rotatory dispersion* in quartz is of very nearly the same magnitude as that produced by some sugar solutions. This fact is utilized in one type of white-light polarimeter for sugar-testing, a quartz plate being used to counterbalance the rotation produced by the solution under test.

Rotation of the plane of polarization by this agency should not be confused with the rotation produced by the half-wave plate. The two phenomena are entirely distinct. In the case of the half-wave plate, rotation was produced only for a crystal of a certain thickness, and the amount of rotation depended upon the orientation of the plate; in this case, rotation is produced for any thickness and is independent of the orientation of the plate.

Substances which have this rotatory effect upon the plane of vibration of light which traverses them are termed *optically active*. (The subject of optical activity is discussed at greater length in the following chapter.)

From the difference in the amount of rotation of different colors, it follows that if plane-polarized white light is incident upon the lower face of the crystal of quartz, the light which is transmitted by the analyzing Nicol will be colored, since this Nicol will extinguish completely only those vibrations which are perpendicular to its transmitting plane. It follows, further, that if the Nicol is rotated through 90 degrees, all the components of the white light which were before extinguished will now be transmitted and vice versa, and hence that the rotation through 90 degrees will cause the color to change to the complement of the first color.

Experiment XXI. Verify the above predictions by setting the polariscope so that light from the sky falls upon it in the manner indicated in Fig. 84. To show that the colors transmitted by the analyzer in planes 90 degrees apart are complementary, it is best to replace the Nicol by a thick crystal of Iceland spar, or some other form of double-image prism, such as a Wollaston, so that both of the colors to be compared may be transmitted simultaneously. As the analyzer is rotated the overlapping portions of the image will be found to maintain the color of white light, while the opposite, non-overlapping portions will be of complementary colors.

It has been found that there are two kinds of quartz, one of which produces rotation to the right, the other to the left. The difference in optical behavior is paralleled by a difference in crystalline structure which makes it possible to distinguish the so-called *right-hand* from the *left-hand* quartz crystals without actually making the optical test. Another example of two chemically identical substances which rotate the plane of polarization in opposite directions is found in solutions of two sugars, glucose and fructose.

CHAPTER XII

ROTATION OF THE PLANE OF POLARIZATION

Optical activity. In some of the experiments described in the preceding chapter it was observed that upon the passage of plane-polarized light through certain substances the plane of polarization underwent a rotation. Such substances were termed *optically active.* These optically active substances comprise solids, liquids (including solutions), and vapors. Rotations thus produced are sometimes in one direction and sometimes in the other. If the direction of rotation of the plane of polarization bears the same relation to the direction of propagation of the light as rotation bears to translation in a "right-handed" screw, the substance in question is said to be *dextrorotatory*, or right-handed; otherwise it is said to be *levorotatory*, or left-handed.

About thirty solids are known which are optically active. Most of these are crystals. In such crystals rotation is produced only when the light passes through them in a certain direction with reference to their optic axes. The most common example is quartz. In many crystals a rotation undoubtedly exists which it is impossible to observe because the rotatory effects are masked by the double-refraction effects. There are a very few amorphous solids which are optically active, and in these the phenomenon is, of course, independent of the direction in which light traverses them.

Over seven hundred liquids and solutions are known which possess optical activity. All are organic compounds or solutions of organic compounds. Some substances which possess optical activity in the solid state retain that characteristic when melted or in solution. It is customary to associate the optical activity of this type of substance with the arrangement of the atoms in the molecule, and that of the type of substance which loses its activity when melted or in solution, with the geometrical arrangement of the molecules in the crystal. It is with the former type that we shall be chiefly concerned.

The amount by which the plane of polarization is rotated when plane-polarized light traverses an optically active substance is found to depend upon a number of factors, the most important of which are thickness, temperature, wave length of the light used, and, for solutions, in addition to the foregoing, concentration. The magnitude of the rotation does not bear any very simple relation to either wave-length or temperature. We shall eliminate any consideration of variation of these factors in our work by maintaining the temperature constant at 20° C. and using always the same kind of monochromatic light for our measurements; namely, that of incandescent sodium vapor, of wave-length $\lambda = 5893$ angstrom units.

Specific rotation. Rotation is found to be directly proportional to the thickness of the substance traversed, and, in the case of solutions, proportional (though only to a first degree of approximation) to concentration. Hence for a given thickness of a certain solution at a definite concentration the rotation should be always the same. The rotation imparted by a ten-centimeter column of a solution containing in every cubic centimeter one gram of optically active solute is termed the *specific rotation* of the solute. It is, of course, necessary that the solvent be optically inactive and that the limitations imposed on temperature and wave-length of light used be observed.

It is seldom possible to find a solvent which will admit of as high a solution concentration as is prescribed in the above definition. It is, moreover, often convenient to carry out measurements on a column of the solution which is of some other length than the prescribed ten centimeters. It is therefore the practice to determine the rotation a, produced by a column l decimeters long, of a solution containing c grams of solute in each hundred cubic centimeters, and then to define specific rotation α by the equation

$$\alpha = \frac{100\,a}{cl}. \tag{1}$$

This will readily be seen to be the equivalent of the preceding definition, on the assumption that a is proportional both to l and to c. This assumption is not strictly true in the case of c. It is found experimentally that slightly different values of α result when concentration varies between wide limits. Proportionality may be assumed, however, as a first approximation, and is sufficiently near the truth for work of any but the highest precision.

Sugar-testing. *Direct method.* Among organic compounds possessing optical activity when in solution are the various types of sugars. Sucrose, or cane sugar, being the most common variety commercially, is often tested for purity by its optical activity. The specific rotation of pure cane sugar has been very carefully determined and may be considered a known quantity. Its value is $\alpha = 66°.50$ at 20° C. for sodium light. It is therefore evident that a careful determination of the rotation a, produced by a column of solution of known length l, should yield a value for c by application of equation (1). Hence the number of grams of pure sugar per hundred cubic centimeters of solution becomes measurable.

If, now, the amount of substance (sugar plus impurities) originally placed in each hundred cubic centimeters of solution is known, a comparison of this with the value of c will enable the per-cent purity of the original substance to be determined. This method of measuring the purity of a sugar sample is sometimes termed the *direct method.*

Invert method. A little consideration will show that the preceding method is reliable only when the impurities are *optically inactive.* If, as is frequently the case, the impurities in a sample of cane sugar consist partly of other varieties of sugars which are also optically active, then the foregoing test will yield very uncertain information.

In order to secure reliable information concerning the purity of a cane-sugar sample under these circumstances, a modification of the foregoing method, commonly called the *invert method*, is used. It consists of bringing about a chemical reaction in the solution which affects only the cane sugar, leaving other impurities, of whatever nature, unaffected. The alteration which this reaction causes in the rotatory power of the solution is then taken as a measure of the amount of pure cane sugar present.

Nature of inversion process. The reaction consists of a transformation of the cane sugar (sucrose) into a mixture of equal portions of glucose and fructose, in accordance with the following equation:

$$\underset{\substack{\text{sucrose}\\342}}{C_{12}H_{22}O_{11}} + \underset{\substack{\text{water}\\18}}{H_2O} = \underbrace{\underset{\substack{\text{glucose}\\180}}{C_6H_{12}O_6} + \underset{\substack{\text{fructose}\\180}}{C_6H_{12}O_6}}_{\text{invert sugar (360)}} \qquad (2)$$

This resulting mixture of equal portions of glucose and fructose is usually termed *invert sugar*. It will be seen from the equation that each part of sucrose is converted into $\frac{360}{342}$ parts of invert sugar. The specific rotations of glucose and fructose are 52°.5 and $-92°.5$ respectively. The minus sign indicates that the rotation produced by fructose is in the opposite direction from that produced by glucose (glucose, being "right-handed," is often termed *dextrose*, and fructose, being "left-handed," is often called *levulose*). The specific rotation of the mixture is, then,

$$\frac{52°.5 - 92°.5}{2} = -20°.0.$$

Since the concentration of this mixture when it is produced by inversion of sucrose is $\frac{360}{342}$ times that of the sucrose, a normal solution of sucrose (1 g. of sucrose per cubic centimeter of solution) will produce a concentration of invert sugar such as to give a rotation of $\frac{360}{342}(-20°.0) = -21°.05$. The total change in the rotatory power of such a solution upon inversion is therefore $66°.50 + 21°.05 = 87°.55$.

These facts may be described by the following equations: Let $-a'$ be the observed rotation of the invert-sugar solution, and β the rotatory power of such optically active impurities as are present. Then, from equation (1), for the original solution,

$$a = \frac{cl}{100} 66°.50 + \beta,$$

and, for the invert solution,

$$-a' = \frac{cl}{100}(-21°.05) + \beta; \qquad (3)$$

whence, subtracting,

$$a + a' = \frac{cl}{100} 87°.55 \quad \text{or} \quad c = \frac{100(a + a')}{l \cdot 87°.55}. \qquad (4)$$

Inversion may be effected by adding concentrated hydrochloric acid to a cane-sugar solution and allowing the mixture to stand for a day at room temperature. The process may be greatly accelerated by heating, but there is involved some danger of affecting other sugars present as impurities. For particularly delicate work a specially prepared substance known as *invertase* is used in place of the hydrochloric acid. The danger of causing a change in the optical activity of the impurities by the inversion process is much less when this agent is used.

EXPERIMENT XII

PURITY OF SUGAR TESTED BY POLARIMETER

Preparation of solutions. Three sugar solutions are to be prepared for the test: (1) a "standard solution," in which the purest sugar obtainable will be used; (2) a solution of cane sugar of unknown purity for measurement by the "direct method"; (3) a solution of invert sugar, prepared from (2), which, used in connection with (2) in the invert method, will give a second determination, of greater reliability, of the purity of the sample used. The sugars to be tested should have been pulverized by mortar and pestle, and dried in a desiccator for not less than twelve hours.

Standard solution. The purest cane sugar readily obtainable is rock candy. Weigh in a watch glass in an analytical balance about 26 g. of this, the sugar having been thoroughly pulverized and dried beforehand. Weighings should be made to the nearest milligram. Every effort should be made to weigh as rapidly as possible in order to minimize the error introduced by absorption of moisture from the atmosphere by the dried sugar. Wash the weighed sample into a clean graduated flask, dilute with distilled water to exactly 100 cc., and shake thoroughly. Keep this solution carefully covered, in order to avoid change of concentration by evaporation.

Cane-sugar solution. Proceed with the sample of impure cane sugar as specified above for the pure sugar. If impurities of the sample are of such a nature as will cause dark color or serious cloudiness in the resulting solution, it will be necessary to clarify the solution by the method described below. In this case do not dilute. Use only enough water to dissolve the sample.

Invert-sugar solution. The following method of inversion of cane sugar, while open to some objections, is in common use. To 50 cc. of the solution of cane sugar prepared as directed above add 5 cc. of a 38.8 per cent solution of hydrochloric acid ("concentrated" HCl, specific gravity $= 1.2$). Add the acid little by little while rotating the flask, in order to mix thoroughly and to secure uniformity of action. Set the mixture aside for twenty-four hours at a temperature not below 20° C. If the temperature is above 25° C., the inversion will be complete in twelve hours. Dilute to 100 cc.

Clarification. The clarification of dark or cloudy solutions is attended with considerable difficulty. The clarification process itself is easily effected; but as a result of the precipitation of the impurities, the process introduces an alteration of the concentration of the solution for which it is difficult to make a reliable correction. The following method is applicable only to mildly turbid solutions, but it has the advantage that it produces for such solutions only a negligible concentration error.

The reagent (alumina cream) may be prepared as follows: Divide into two unequal portions a saturated solution of alum in water. Add a slight excess of ammonium hydroxide to the larger portion, and then add by degrees the remaining alum solution until a faintly acid reaction is secured. The reagent is now ready for use.

To apply it, add minute portions to the turbid solution. Avoid an excess, *using only the minimum amount necessary to clarify the solution.* Only experience will enable the experimenter to judge when the solution is sufficiently clear. Any solution darker than a light straw color will be unsuited for polarimetric measurement. It may be found necessary on occasion to use an amount of alumina cream equal to or even greater than the original volume of the solution. It is for this reason that it is unwise to use more water than is necessary to dissolve the sample when clarification is contemplated. It will be necessary, of course, to take account of the dilution in subsequent calculations.

After clarification dilute to 100 cc. and filter. During filtration cover with a glass plate, to minimize evaporation. Measurements may now be taken on the clarified solution as described below.

Equipment for sugar preparations. The following equipment will be used in the preparation of sugar samples. It is convenient to arrange for this purpose a special cabinet in which the equipment may be kept and reserved for this particular work. In addition to the equipment listed below, an analytical balance will be found necessary.

4 sixteen-ounce bottles, numbered, for sugar samples.
4 eight-ounce bottles, numbered, for cane-sugar solutions.
4 eight-ounce bottles, numbered, for invert-sugar solutions.
2 graduates, 100 cc.
2 graduates, 25 cc.
2 stirring-rods.
2 beakers, 250 cc.
4 evaporating-dishes, 100 cc.
1 desiccator.
1 thermometer.
1 glass funnel.
1 ring stand.
Concentrated hydrochloric acid.
Alum.
Ammonium hydroxide.
Litmus paper.
Filter paper.
Extra bottles.

Apparatus. Measurements of rotation of the plane of polarization are invariably made with an instrument called the *polarimeter.* In its simplest form the polarimeter consists of two Nicol prisms so mounted that a column of the liquid under investigation may be placed between them. The analyzing Nicol is so arranged that the angle through which it is rotated may be read on a graduated circle. If the polarimeter is intended solely for sugar analysis, the graduated circle is calibrated in terms of percentages of purity in place of degrees of arc, and the instrument is called a *saccharimeter.* The accuracy attainable with the simplest types of polarimeter is not very great, for when the analyzer is set for extinction it may be rotated several degrees each way from a mean position without affecting the appearance of the field.

The methods employed in increasing the attainable accuracy of setting are different in different forms of the instrument. That most frequently used is the introduction, directly behind the polarizer, of some device which has the effect of dividing the field of view into two or more parts whose planes of polarization make a small angle with each other. Thus in the so-called *Laurent*, or half-shade, polarimeter there is introduced a semicircular half-wave plate whose optic axis makes a small angle with the plane of polarization of the light. It was observed in the preceding chapter that the effect of introducing a half-wave plate into a beam of plane-polarized monochromatic light is to rotate the plane of polarization. The result is that if the analyzing Nicol is set for extinction of half the field, the other half is found to possess appreciable intensity (Fig. 89, *a*). A small rotation of the analyzer reverses the intensities of the portions of the field (Fig. 89, *b*). For an intermediate position the two halves will appear of equal intensity (Fig. 89, *c*). The eye can judge far more accurately the positions of equal illumination of the two halves of the field than it can the position of total extinction of the whole field. Hence the analyzer can be set to this position of equal intensities with considerable precision as compared with the simpler type of polarimeter. It should be remarked that the chief objection to the Laurent polarimeter lies in the fact that the half-wave plate limits the instrument to use with one particular wave-length of monochromatic light.

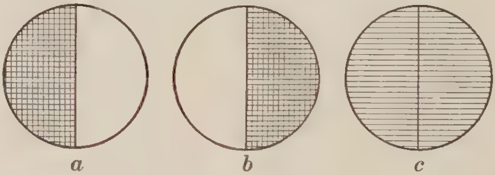

FIG. 89. Field of view of Laurent polarimeter

Fig. 90 illustrates the arrangement of the different parts of the Laurent polarimeter. The polarizer P, the analyzer A, the half-wave plate h, and the graduated circle C will be readily distinguished. F represents a crystal of potassium bichromate, often permanently mounted, which has as its object the filtering of white light to a monochromatism approximating that of sodium light. This is often a desirable feature, since a sodium source is not always convenient to secure or to use. It may be noted in this connection that the use of a permanently mounted crystal is not approved by the Bureau of Standards, which recommends an absorption cell containing a solution of potassium bichromate which is the equivalent of a 15-millimeter thickness of a 6 per cent solution. With the filter in place, white light may be used for observations, but the wave-length of sodium light is used in calculations. The Bureau rather urgently recommends the adoption of the green radiation from mercury ($\lambda = 5461$) in place of sodium light. In this case a different value for α from the one here given would be used. S represents the tube which contains the solution under test, and T a short-focus telescope which is focused upon the edge of the half-wave plate in order that the line of division between the two halves of the field may be made as sharp as possible.

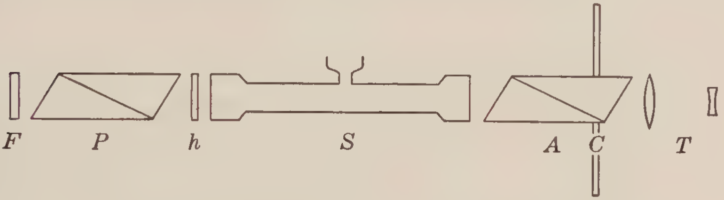

FIG. 90. Optical system of Laurent polarimeter

Adjustments. If a sodium-light source is used, a light filter will be unnecessary; and in case a filtering-crystal is mounted on the instrument, it should be removed. If white light is used, retain the crystal in place or substitute for it an appropriate thickness of the solution mentioned above.

The polarizing Nicol may be rotated through a small angle by a lever which moves over a short, graduated arc. Moving this lever changes the sensitivity of the instrument. The best position of the polarizer can be determined only by experiment. In general the smaller the angle which the plane of polarization of the transmitted light makes with the axis of the semicircular half-wave plate, the higher will be the sensitivity of the instrument. The greater the intensity of the source, the smaller may this angle be made. Hence the desirability of a very intense source.

Measure carefully the length of the tube used. Standard tube lengths for the polarimeter are 1 and 2 dm. and for the saccharimeter .947 and 1.894 dm.

Fill the tube with distilled water, place it in the instrument, and focus carefully on the dividing-line between the two halves of the field. Determine the reading, taking the mean of several settings. It will probably be found better in this and in succeeding observations to read the position of equal darkness rather than that of equal lightness of the two halves of the field.

Filling the polarimeter tube. The solution upon which measurements are to be taken should be brought to a temperature of 20° C. The tube should be initially clean and dry, and should be rinsed two or three times with the solution before being finally filled. During this rinsing, the bottle containing the solution should be given a rotary motion to stir its contents, thus bringing the solution to a uniform density. In attaching the end, care should be taken to avoid the presence of air bubbles in the tube. Avoid also the application of any considerable force in screwing on the ends of the tube. Glass, when placed under strain, becomes optically active. Straining the cover glasses of the tube is hence liable to introduce error into the subsequent readings. If rubber or leather washers are used, *do not* put them between the tube and the cover glass, but between the cover glass and the brass screw end.

During the filling and subsequent manipulations every precaution should be taken to maintain a constant temperature in the solution under test. One of the most frequent sources of error lies in holding the tube in the hand in such a way that the heat of the hand warms it. A polarimeter tube provided with a water jacket is very desirable, especially for the invert solution.

Measurements. *The specific rotation of cane sugar.* Fill the tube with the "standard solution." Take the mean of several readings for the new position of the analyzer, and record the amount of rotation which the solution produces in the plane of polarization. Substitute in equation (1) to find a value for α, the specific rotation of cane sugar. This should come very close to the commonly accepted value of 66°.50 at 20° C. If it is impracticable to maintain a temperature of exactly 20°, the constant 66°.50 should be corrected by subtracting 0°.02 for every degree above 20.

Testing purity of cane-sugar sample by direct method. The direct method is essentially a comparison method. It is based primarily on the assumption that, other conditions being the same, the amount of rotation is proportional to the concentration of the solution. It is vitally important to take measures to insure that other conditions, particularly temperature, *are* the same for the two measurements.

Repeat the measurements of the preceding section, using the solution of unknown purity. Calculate the value of c of equation (1), using 66°.50 for the value of α (see the preceding page for temperature correction). Compare this value of c (the weight of pure cane sugar per hundred cubic centimeters of solution) with the known quantity of substance (sugar plus impurities) originally placed in solution, and deduce the per-cent purity of the original sample.

Testing purity of cane-sugar sample by invert method. The sources of error in the preceding method have already been pointed out. They are obviated by measuring the change in optical activity of the solution when it is subjected to the inversion process. Fill the tube with the invert solution. Take the same observations as have been prescribed for the other two samples. It is particularly necessary to have this solution at the proper temperature, since the error introduced by improper temperature is nearly fifteen times as great for a solution of invert sugar as for one of cane sugar.* It will be observed that the plane of polarization is rotated by the invert solution in a direction opposite to that effected by the cane-sugar solution. The amount of rotation thus observed must be multiplied by 2, to allow for the fact that in the process of inversion the solution was diluted to half its original concentration. Call the resulting angle a'. Calculate c by equations (4).

Compare this value of c (the weight of pure cane sugar per hundred cubic centimeters of solution) with the known quantity of substance originally placed in solution, and deduce the per-cent purity of the original sample. It is quite probable that this value of the purity will not agree with that given by the direct method. Of the two the latter value is the more reliable, since it eliminates the effect of optically active impurities.

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* The temperature correction in case of invert sugar is − 0°.304 for every degree in temperature above 20° C.

CHAPTER XIII

ELLIPTICALLY POLARIZED LIGHT

Production and measurement. In Chapter XI we considered briefly the nature of elliptically polarized light and methods for producing and measuring it. Since measurements of the nature of those involved in the analysis of elliptically polarized light form an important part of some fields of optical research, it is considered wise to devote more attention to these principles than was feasible in connection with Chapter XI.

It is possible to measure roughly the characteristics of light known to be elliptically polarized, by observing the positions and estimating the relative intensities of the two minima and maxima occurring with each revolution of the analyzer by means of which the light is being investigated. It will readily be appreciated, however, that this method is of doubtful precision. For this reason other methods are invariably used in actual practice. The principle of one of these will be most readily grasped by first considering in greater detail the nature of elliptically polarized light.

Two methods of producing elliptical polarization of light were mentioned in Chapter XI. One of these involved a difference of 90 degrees in phase between the components (production by a quarter-wave plate), and the other involved a phase difference other than 90 degrees (production by a plate of some thickness other than that of the quarter-wave plate). Either method produces an elliptical vibration. The distinction between the two is that, whereas in the first case one axis of the resultant elliptical vibration is parallel to the optic axis of the plate, in the second case this is not true, the position of the ellipse axes being affected by the thickness of the plate as well as by its position.

But from considerations of elementary mechanics an elliptical vibration, no matter how it is produced, may always be regarded as the resultant of two simple harmonic motions of equal period, unequal amplitudes, and a phase difference of 90 degrees, compounded at right angles with one another. If an alteration of 90 degrees is effected in the phase of one of these motions, the two will then combine into a linear instead of an elliptic motion.

Such a phase alteration may be produced in the case of elliptically polarized light by introducing into the path of the beam a quarter-wave plate with its optic axis parallel to one axis of the ellipse. This reduction of elliptically polarized light to plane-polarized light constitutes one of the standard methods of measurement in this field.

Equation of an elliptical vibration. The situation may perhaps be somewhat clarified by considering the equations of motion of a particle which is subject to the conditions that were outlined above. Suppose we have given a beam of plane-polarized light whose vibrations take place in the plane of *OC* (Fig. 91). Let *OC* represent the amplitude of the vibration. Suppose, further, that a plane-parallel plate of a doubly refracting crystal (not cut perpendicularly to the optic axis) be introduced into the path of the beam in such a way that its faces are perpendicular to the beam, while its principal plane has the direction *OX*. The incident vibration *OC* will be divided by the plate into two components *OF* ($= a$) and *OG* ($= b$), which will be equal respectively to $OC \cos\theta$ and $OC \sin\theta$, if θ represents the angle *COF*. These two component vibrations travel through the plate with different velocities; hence when they emerge, though they will still be vibrating parallel to *OF* and *OG*, one will be in advance of the other by an amount which depends upon the

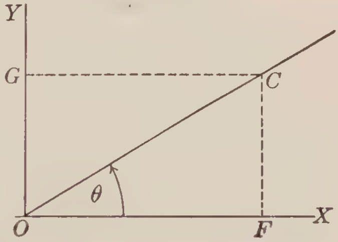

FIG. 91. Resolution of light vibrations by crystal plate

thickness of the crystalline plate and the difference of the velocities of the two components in it. If we take $OC \cos 2\pi \frac{t}{T}$ to represent the original vibration, and δ to be the difference in the optical paths of the two components as they traverse the crystal, the vibrations of the two components will be represented by

$$x = a \cos 2\pi \frac{t}{T} \quad \text{and} \quad y = b \cos 2\pi \left(\frac{t}{T} + \frac{\delta}{\lambda}\right) \tag{1}$$

respectively. These vibrations reunite into a single one. The path which any vibrating particle pursues may be determined by eliminating t from equations (1). This is recommended to the student as an exercise. The result is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy \cos 2\pi \frac{\delta}{\lambda}}{ab} = \sin^2 2\pi \frac{\delta}{\lambda}. \tag{2}$$

Properties of an elliptical vibration. An examination of equation (2) for maxima and minima of the values of both x and y will give $x = \pm a$ and $y = \pm b$ respectively. Equation (2) is in fact the equation of an ellipse that is inscribed in a rectangle whose sides are $2a$ and $2b$ respectively. The position of the ellipse in the rectangle (that is, the orientation of its axes) depends manifestly upon the value of δ. Two cases are of special interest by reason of their simplicity: First, when $\delta = \frac{\lambda}{2}$, we have the condition for the half-wave plate. In this case $\sin 2\pi \frac{\delta}{\lambda} = 0$, $\cos 2\pi \frac{\delta}{\lambda} = -1$, and the path of the particle is given by

$$\frac{x}{a} + \frac{y}{b} = 0; \tag{3}$$

that is, the emergent light is again plane-polarized. If, in addition, $\theta = 45°$, then $a = b$, and the plane of polarization is at right angles to that of the incident light. This is the condition for the half-wave plate which is discussed in Chapter XI.

Second, when $\delta = \frac{\lambda}{4}$, we have the condition for the quarter-wave plate. In this case $\sin 2\pi\frac{\delta}{\lambda} = 1$, $\cos 2\pi\frac{\delta}{\lambda} = 0$, and the equation for the path reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \tag{4}$$

that is, the path of vibration in the emergent light is an ellipse one of whose axes coincides in direction with the principal plane of the crystalline plate. If, in addition, $\theta = 45°$ and hence $a = b$, the equation becomes

$$x^2 + y^2 = a^2; \tag{5}$$

that is, the path of the vibration is a circle, which is the condition discussed for the quarter-wave plate in Chapter XI.

But if δ possesses neither of the above values nor any multiple thereof, then equation (2) cannot be so readily simplified, though it still represents the equation of an ellipse. This would be the situation inevitably encountered for an arbitrarily chosen plate. While it would be theoretically possible to rotate the coördinate axes by familiar methods of analytical geometry, and thus derive the constants of the ellipse in terms of the initial conditions, the graphical analysis outlined in the following section will probably be found preferable.

Method of measurement. A complete identification of the ellipse of vibration requires the evaluation of two independent constants. The experimental analysis of elliptically polarized light consists therefore of two distinct steps: first, the determination of the orientation of the axes of the ellipse, and, second, either the ratio of the two axes or the phase lag (δ).

For the first of the latter two the quarter-wave plate is used as described above, in conjunction with a Nicol prism. The position of the quarter-wave plate gives the orientation of the axes, and the position of the Nicol the ratio of the axes. For the second some form of wave plate of variable thickness is indispensable. The Babinet compensator (see page 153) is the type in most common use. This is used in conjunction with a Nicol, the phase difference (δ) being determined by the thickness of the compensator plate which is necessary to restore a plane-polarized beam to its initial condition, while the orientation of the axes is given by the position of the Nicol, as before.

Only the first method, that involving the use of the quarter-wave plate, will be described in this chapter. Those to whom a Babinet compensator is available will find directions for its use in Appendix II.

Let OH (Fig. 92) represent direction and amplitude of the plane-polarized light that is incident upon the plate, which may conveniently be a thin piece of mica. Let OA and OB be the planes of transmission of the mica and let EEE represent the resulting elliptical path of the vibration which it is desired to analyze. Upon the introduction of a quarter-wave plate properly oriented, this elliptical vibration becomes transformed into a plane vibration which may be represented by the line OC. To extinguish this vibration the analyzer must be turned so that its plane of transmission is perpendicular to OC; that is, so that it has the direction OB'. The proper orientation of the quarter-wave plate is, in fact, found by rotating it and the analyzer simultaneously until they have such positions that the light is completely extinguished. The axis of the quarter-wave plate is then parallel to either OE or OF, and the plane of transmission of the analyzer is represented by OB'. Hence the position of the principal axes of the ellipse of vibration is identified by the position of the optic axis of the quarter-wave plate, and the ratio of the axes of the ellipse is given by the tangent

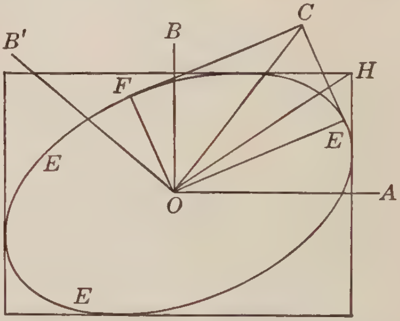

FIG. 92. Application of quarter-wave plate to the analysis of elliptically polarized light

of the angle between this optic axis and the plane of transmission of the analyzer; that is,

$$\tan B'OF = \tan COE = \frac{EC}{OE} = \frac{OF}{OE}. \tag{6}$$

EXPERIMENT XIII

ANALYSIS OF ELLIPTICALLY POLARIZED LIGHT

Apparatus. The most convenient instrument for determining the nature of elliptically polarized light is an ordinary spectrometer to which have been added a pair of Nicols and three graduated circles. The circles should be mounted on short tubes, with their planes perpendicular to the length of the tubes. Two of these tubes should be made to fit over the telescope and collimator at their objective ends, while the third fits over the eye end of the telescope. Into each of these tubes fits a collar which may be rotated within the tube. These collars carry indices by which their positions may be read on the graduated circles upon the tubes.

To fit within the collar at the objective end of the collimator there should be provided a double-image prism (a Wollaston prism is very convenient) and a Nicol prism (the polarizer). Another Nicol, the analyzer, is to be permanently mounted in the collar at the eyepiece end of the telescope. It is convenient to place this between the lenses of the eyepiece itself, though it can be mounted either in front of or behind the eyepiece. A second eyepiece, of the Gauss type, will be needed to replace this one in the preliminary adjustments of the instrument and in the measurements of the next experiment. In the collar at the objective end of the telescope a quarter-wave plate is to be placed.

In making measurements a monochromatic source of illumination should be used, of as great intensity as is securable. The oxy-gas flame with the hard-glass rod is recommended. For the adjustments a white-light source is the most convenient.

Adjustments. The tubes carrying the graduated circles having been set in place upon the collimator and telescope of the spectrometer, preferably with their zero points upward, the Nicols must be so mounted in these tubes that their planes of transmission are either vertical or horizontal when the circles read zero.

To attain this, place in the tube at the end of the collimator the double-image prism. Using the Gauss eyepiece, or some auxiliary eyepiece not containing a Nicol, two images of the slit will be seen. Rotate the double-image prism until these two images are superposed in the center of the field, one projecting above the area in which they are superposed, the other below it. If the slit of the collimator is vertical, then, when the images overlap as described, the vibrations of the light composing one of the images will be vertical, while those of the other will be horizontal.

Replace the auxiliary eyepiece by the one containing the Nicol, and, holding its index at zero, turn the Nicol until it entirely extinguishes one of the images of the slit. The double-image prism should then be replaced by another Nicol, the index of the circle on the collimator should be set to zero, and the Nicol should be rotated in the collar until the other image of the slit is extinguished. The plane of transmission of the Nicol on the collimator is then either vertical or horizontal when the index by which its position is read stands at zero. If the plane along which the two halves of the Nicol are cemented together is vertical, the transmitted vibrations are horizontal and vice versa. This adjustment can also be made by setting the analyzer so as to extinguish the light reflected from a vertical plate of glass at the angle of complete polarization. When this takes place, the plane of transmission of the analyzer is horizontal.

Substitute the monochromatic for the white-light source. Mount the quarter-wave plate in place at the objective end of the telescope and, holding the index at zero, turn the plate until the field is dark, the Nicols being "crossed" and their indices at zero. The planes of transmission of the quarter-wave plate are then respectively vertical and horizontal.

Measurements. Introduce a thin plate of mica between the collimator and telescope, by inspection setting it normal to the line of sight. The field, which was originally dark, will brighten. It will not be found possible to produce complete extinction again by merely turning the analyzer. There exists, however, a position of the quarter-wave plate and the analyzer for which extinction can be secured. To find it, proceed as follows:

By turning the analyzer locate approximately a position of minimum intensity. Turn the quarter-wave plate through a small angle, say 5 degrees, and again search for the position of minimum intensity. When this is found, the intensity will be either greater or less than it was before. If it is greater, the plate was turned in the wrong direction; if less, in the right direction. Continue the search, repeating these manipulations until the position of the quarter-wave plate and analyzer is found for which extinction is produced. If these positions are represented by rotations of α and β respectively, from the initial positions, then the principal axes of the elliptical vibration form an angle α with the horizontal and vertical respectively, and the ratio of the amplitudes of the two component vibrations is

$$\tan(\beta - \alpha).$$

CHAPTER XIV

THE REFLECTION OF POLARIZED LIGHT

Directions of reflected and of refracted light. When a beam of light encounters the boundary between two transparent isotropic media, a portion of it is reflected and a portion is transmitted into the second medium. The so-called law of reflection,

$$\text{Angle of reflection} = \text{angle of incidence}, \tag{1}$$

and the law of refraction,

$$\mu_1 \sin i = \mu_2 \sin r, \tag{2}$$

are generalizations of the results of all experimental investigations concerning the directions of the reflected and refracted rays. These laws are so firmly established and are to such an extent a matter of common observation that we are inclined to forget that at least until recent years their theoretical basis has been decidedly insecure.

Intensities of reflected and of refracted light. The question as to what fraction of the incident light is reflected and what fraction is transmitted is, from some standpoints, of still greater interest. Experimental investigations on this aspect of the laws of reflection and refraction have made it possible, in a number of important instances, to render decisions between rival theories of the nature of light. The relations which have been most successful in withstanding these tests may be most conveniently stated in the form of what have come to be known as "Fresnel's equations" for the reflection and refraction of polarized light.

If a plane-polarized beam of light consisting of vibrations of amplitude a, perpendicular to the plane of incidence,* is incident upon the surface of an isotropic transparent substance, the amplitude b of the reflected light will be

---

* In this connection see the footnote on page 143.

$$b = -a\frac{\sin(i-r)}{\sin(i+r)}, \tag{3}$$

and the amplitude c of the refracted light will be

$$c = a\frac{2\sin r\cos i}{\sin(i+r)}. \tag{4}$$

But for a beam consisting of vibrations *in* the plane of incidence, the corresponding relations are, for the reflected part,

$$b_1 = a_1\frac{\tan(i-r)}{\tan(i+r)}, \tag{5}$$

and, for the refracted part,

$$c_1 = a_1\frac{2\sin r\cos i}{\sin(i+r)\cos(i-r)}. \tag{6}$$

The intensities will, of course, be proportional to the squares of the amplitudes as given by equations (3), (4), (5), and (6). In case the incident light is so polarized that its vibrations occur neither in nor perpendicularly to the plane of incidence, or where the incident light is entirely unpolarized, the component vibrations must be considered separately and the results compounded.

Bearing of Fresnel's equations upon light theories. It is interesting to observe that Fresnel's equations were originally an outgrowth of the old *elastic solid* theory of the ether and for a time lent considerable weight to the hypotheses which constituted this theory. Their strength lay, however, not in their theoretical background, but in the fact that the equations themselves gave an excellent description of experimental results. It was therefore inevitable that their form should remain essentially unchanged, even though there was substituted for their old theoretical foundation a new and entirely different one, the electromagnetic theory of light.

It is not feasible to carry out, within the limitations of this text, a derivation of Fresnel's equations from first principles.

Such derivation on the modern theoretical basis will be found in any treatise on the electromagnetic theory. Houstoun's "A Treatise on Light" presents it in what is probably the simplest possible way. Students who are interested in the historical aspect of these equations will find their derivation on the basis of the elastic-solid theory in many of the standard works on light, those in English being notably Preston's "Theory of Light" and Edser's "Light for Students." Since we shall concern ourselves mainly with the laws of reflection, our chief attention will be devoted to an investigation of the consequences of equations (3) and (5).

It is perhaps unnecessary to point out that the mere form of these equations suggests that the reflected light is affected by the nature of the medium from whose surface it is reflected. For the elimination of r from them by means of equation (2) would involve introducing the term $\frac{\mu_2}{\mu_1}$, the index of the second medium with respect to the first. It follows that the reflected light must have penetrated a certain distance into the second medium in the process of reflection. Investigation has indicated that the depth of this penetration is only a small fraction of a wave-length of light.

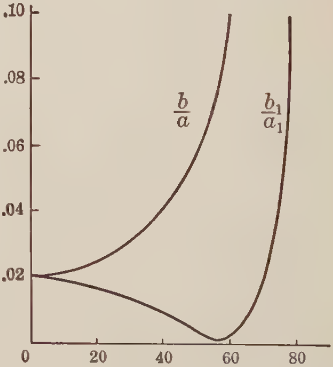

FIG. 92 A. Intensities of reflected light as a function of angle of incidence

Consequences of Fresnel's equations. Equations (3)–(6) are represented graphically in Figs. 92 A and 92 B for limited ranges of the angle of incidence. In the case of equation (3), the light vibrations being perpendicular to the plane of incidence, if we assume the refractive index of the second medium to be greater than that of the first, then $\mu\left(=\frac{\mu_2}{\mu_1}\right)>1$, and $i>r$. Hence the amplitude of the reflected wave is opposite in sign to that of the incident wave; that is, there is a phase-change of 180 degrees at reflection, whatever the angle of incidence. In absolute value $b < a$, becoming equal to a only when $i = 90°$. At normal incidence, $i = 0$, and, by equation (2), $r = 0$. Hence the right side of equation (3) becomes indeterminate. It may be evaluated by proceeding to the limit and substituting $i = \mu r$; whence

$$b = -a\frac{\mu - 1}{\mu + 1}. \tag{7}$$

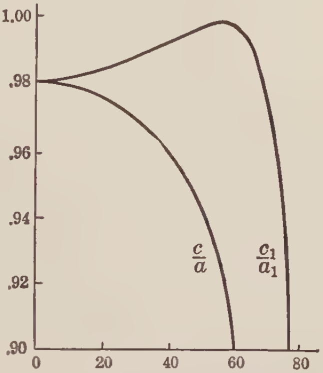

FIG. 92 B. Intensities of refracted light as a function of angle of incidence

For the case of equation (5),—the light vibrations being in the plane of incidence,—if, as before, we assume the index of the second medium to be greater than that of the first, then $\mu > 1$, and $i > r$. If, also, $i + r < 90°$, the amplitude of the reflected wave is the same in sign as that of the incident wave; that is, there is in this case no phase-change at reflection. If, however, $i + r > 90°$, no such phase-change occurs. When $i + r = 90°$, $b_1 = 0$; that is, no light is reflected. For this case $\sin r = \cos i$; and hence, by equation (2),

$$\tan i = \mu. \tag{8}$$

This will be recognized as Brewster's law (p. 141), which is thus seen to be one of the consequences of Fresnel's equations. The student may show that in this case the amplitude of vibration in the refracted ray is

$$c_1 = \frac{a_1}{\mu}. \tag{9}$$

By the same type of reasoning as led to equation (7), it may be shown that for this case also, at normal incidence,

$$b_1 = -a_1\frac{\mu - 1}{\mu + 1}. \tag{10}$$

Experimental verification of Fresnel's equations. Fresnel's reflection equations have been frequently verified experimentally. One method of testing them is, of course, to make photometric measurements of the relative intensity of incident light and reflected light under the conditions of equations (3) and (5) and to compare the observed with the calculated ratios. Photometric measurements are always troublesome, however, and their precision is seldom high. The reflection equations may be more easily tested in another manner. This method depends upon the fact that if the incident light is plane-polarized, but neither in nor perpendicularly to the plane of incidence, the plane of polarization undergoes, in consequence of reflection, a rotation, the amount of which may be predicted by applying equations (3) and (5) in a modified form.

Rotation of plane of polarization by reflection. Let the plane of polarization (*not* the plane of vibration; see footnote, p. 143) of the incident light make an angle θ with the plane of incidence.

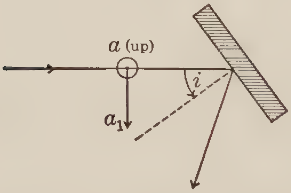

FIG. 93. Reflection of plane-polarized light

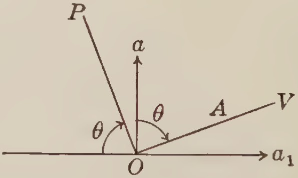

FIG. 94. "Plane of polarization" vs. "plane of vibration"

We must now consider the component vibrations *in* and *perpendicular to* the plane of incidence, calling them for convenience the a_1 and a components respectively.

Fig. 93, in which the plane of the paper is the plane of incidence, shows the position of the glass surface, the direction of propagation of the incident and reflected rays, and the positive directions of the a and the a_1 axes. In Fig. 94 the reader is looking along one of the incident rays. OP represents the plane of polarization, and OV the direction of vibration (since we have adopted Fresnel's assumption that the direction of vibration is perpendicular to the plane of polarization). The plane of incidence would cut the plane of this figure in a horizontal line. Hence a_1 is *in* and a is *perpendicular to* the plane of incidence, as stated above. We may therefore write

$$a_1 = A\sin\theta \quad \text{and} \quad a = A\cos\theta. \tag{11}$$

Applying equations (3) and (5) to these component vibrations,

$$b = -a\frac{\sin(i-r)}{\sin(i+r)}. \tag{12}$$

$$b_1 = -a_1\frac{\tan(i-r)}{\tan(i+r)}. \tag{13}$$

These two components will unite to form a plane-polarized beam whose inclination (θ') to the plane of incidence is given by the relation

$$\tan\theta' = \frac{b_1}{b}; \tag{13'}$$

whence, substituting,

$$\tan\theta' = \frac{a_1}{a}\frac{\cos(i+r)}{\cos(i-r)} = \frac{\cos(i+r)}{\cos(i-r)}\tan\theta. \tag{14}$$

If we know the index of refraction of the glass from the surface of which the reflection takes place, equation (14) may be checked experimentally. To do this give θ and i any desired values, and substitute for r its value calculated from the known refractive index. θ' may now be calculated, and checked by observation.

EXPERIMENT XIV

VERIFICATION OF FRESNEL'S LAWS OF REFLECTION

Apparatus. The same as for Experiment XIII, except that the glass surface upon which measurements are to be made is substituted for the mica sheet and that the quarter-wave plate may be dispensed with.

Adjustments. Cleanliness of the glass surface upon which measurements are to be made is very important in this experiment. It is advisable to clean it by the method described in connection with the preparation of glass surfaces for silvering (see Appendix II), after which it should not be touched with the hand. During measurements frequent cleaning of the surface with a tuft of cotton tied to a glass rod will be found advisable.

The spectrometer having been previously adjusted by the methods of Chapter III, clamp to the prism table the plate upon which measurements are to be made, and set it parallel to the axis of the instrument by the use of a Gauss eyepiece.

Record the reading of the circular scale of the spectrometer when the axis of the telescope is perpendicular to the glass surface. All measurements of angle of incidence are to be referred to this reading.

Monochromatic light of any wave-length for which the refractive index of the plate is known may be used.

Note that the analyzer should be originally set so as to extinguish light polarized in the plane of incidence, and that this should be taken as the zero point to which all measurements of θ' are referred.

Measurements. The measurements consist in the observation of the values of θ' corresponding to a given θ for a series of values of i. For a reason which will appear presently, it is advisable to take values of i at a number of positions, with a regular interval of about 3°, on each side of what is estimated to be the position of complete polarization (this will be approximately $i = 57°$ for crown glass). In measuring i be sure that both Nicols are always set at the same point. Otherwise the displacement of the image of the slit, due to the fact that the Nicol transmits the extraordinary ray, will give erroneous values for i.

The following measurements are typical. For this series the polarizer was set so that $\theta = 45°$.

i	θ' (Observed)	θ' (Calculated)
52° 48′	6° 15′	5° 45′
55° 18′	2° 15′	1° 52′
57° 48′	− 2° 0′	− 2° 2′
60° 18′	− 6° 15′	− 5° 56′
62° 48′	− 10° 0′	− 9° 48′

The surface used in the experiment which gave the above data was that of a crown-glass prism whose index of refraction, measured by the method of minimum deviation, was found to be 1.5106. The index as determined by the method of the succeeding paragraph was 1.5108.

From the series of observations listed above, the value of the refractive index of the plate may be found, in case it is otherwise unknown, for use in calculating the values of θ'. In the remarks leading up to equation (8) it was pointed out that the condition defining the angle of complete polarization was $i + r = 90°$. But from equation (14), $\theta' = 0$ when $i + r = 90°$. Hence if the values of i are plotted as abscissas against the observed values of θ' as ordinates, the point where the curve crosses the axis of abscissas will give to i a value which represents the angle of complete polarization. From this, μ may be found by equation (8).

When μ is known, the value of r corresponding to each value of i may be found by equation (2), and hence the calculated values of θ' may be found by equation (14).

CHAPTER XV

THE SPECTROPHOTOMETER

Comparison of luminous intensities. Comparison of the total visible radiation of two sources is effected by various types of photometer. The use of most of these presupposes that the illuminants compared are of sensibly the same color. This is a condition which in actual practice it is difficult to satisfy. Fortunately comparison of the visual intensities of sources whose colors are widely different is not often necessary, primarily for the reason that any statement of the numerical value of such intensity ratios is liable to be largely meaningless.

The spectrophotometer. A difference in the color of two sources is to be interpreted as a difference in the distribution of intensities across the visible spectrum. Comparisons of sources of different color, although largely meaningless when stated in terms of total visible radiation, become very useful when expressed in terms of a quantitative comparison, wave-length for wave-length, of the intensities of their visible spectra. Instruments for making such comparisons have become known, quite appropriately, as *spectrophotometers.*

There is considerable variety in the types of spectrophotometer at present in use. In essence each instrument is a combination of photometer and spectrometer, and hence in construction must embody the essential features of both. These features are (1) a device for decomposing the illumination from each source into its spectrum; (2) a device for bringing the illumination from these two sources simultaneously into the field of view in such a way as to avoid a line of division between the two portions of the field; (3) a device for bringing the two portions of the field to a photometric match and for deducing the relative intensities.

Types of spectrophotometer differ chiefly in the ways in which these functions are performed. A comparison of the relative merits of the different types of instrument now in use is unnecessary. The most appropriate type would be at least partly determined by the nature of the problem in hand. A type which combines general utility with a high degree of potential precision is the Lemon modification of the Brace spectrophotometer. This instrument will be somewhat briefly described here, the details of its construction and adjustments being more fully set forth in the following laboratory instructions.

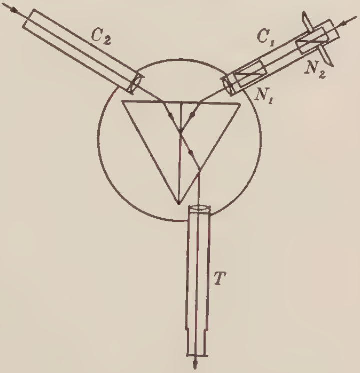

FIG. 95. Path of light in the Brace-Lemon spectrophotometer

The Brace-Lemon spectrophotometer. This instrument is illustrated in plan in Fig. 95. It will be seen to be a modification of the prism spectrometer. One single prism effects the necessary decomposition of illumination from both sources, one of which is placed in front of one collimator (C_2), and the other, the standard, in front of the other collimator (C_1). Light from the two sources is brought simultaneously into the field of view, without an appreciable dividing-line between the two portions, by a special construction in the prism which constitutes the essence of the original Brace instrument, and which will be described presently. The photometric match is secured and the relative intensities are measured by the polarizing system embodied in the Nicols N_1 and N_2. This feature is the modification, introduced by Lemon, of the original Brace design.

The Brace prism. The complete prism is illustrated in perspective in Fig. 96. It presents the appearance of a 60-degree prism of the type commonly used in spectrometry, except that in the plane bisecting the angle A there is a silver strip ss'.

This strip is deposited on the surface of one of the 30-degree prisms, which are afterwards cemented together to form the complete 60-degree prism. Light from the collimator C_2 (Fig. 95) passes through one portion of the prism, and is dispersed in the usual manner. Light from C_1, however, is reflected by the silver strip, thence traveling parallel to the light from C_2. Consideration of the symmetry of the figure will show that even though reflected within the prism, light from C_1 is subject to the same dispersion as that from C_2. The field of vision is seen to be divided into two parts, one representing the intensity of the standard in that portion of the spectrum upon which the telescope is set, and the other representing the intensity of the source under investigation for the same portion of the spectrum. These two portions of the field will in general be of different intensity.

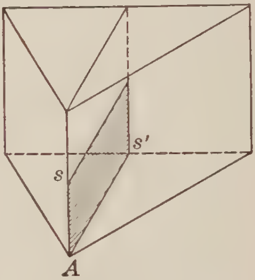

FIG. 96. The Brace prism

Comparison of intensities. The photometric match is effected by setting the Nicol N_2 to a position in which the entire field becomes of uniform shade. Precision in this setting depends almost entirely upon the absence of a dividing-line, whether of overlapping or of separation, between the two portions of the field.

The ratio of the intensities of the sources is deduced from the orientation of the Nicol N_2 in the following manner: When the Nicols are crossed, the intensity of that portion of the field illuminated from collimator C_1 is zero. For any other relative position the intensity will be determined by the component, parallel to the plane of transmission of N_1, of the amplitude of the light vibrations which pass through N_2. This will be proportional to $\sin\theta$, where θ is measured from the crossed position. Intensity, being proportional to the square of the amplitude, is therefore proportional to $\sin^2\theta$.

EXPERIMENT XV

COMPARISON OF SPECTRAL DISTRIBUTIONS OF INTENSITIES OF VARIOUS LIGHT SOURCES

Apparatus. The best standard in comparisons of this type is sunlight. In case sunlight is used, a heliostat and one or more large plane mirrors (preferably mounted in *porte lumière* style) will be necessary to secure a steady beam of light at the place where it is convenient to use the spectrophotometer. Since direct sunlight is often unavailable, some secondary standard which is a fairly faithful imitation of sunlight will always be necessary. The Nernst glower is to be preferred for this purpose, with the gas-filled tungsten incandescent a reasonably satisfactory second choice. An ordinary tungsten lamp on twice its rated voltage approximates sunlight.

For comparison with the standard, an assortment of commercial illuminants will be provided by the instructor. Among these will be tungsten and carbon-filament electric lights, a Welsbach gas mantle, etc. In most cases it will be found convenient to interpose a ground-glass screen between the source and the instrument.

For work with discontinuous spectra a mercury arc or helium discharge tube will be needed. A Gauss eyepiece and a mounted plane-parallel plate will be necessary in making the initial adjustments.

Adjustments. The student is cautioned that the Brace prism is exceedingly fragile. Any adjustment which involves handling the prism (as in the case of some of the adjustments described below) should be made only under the immediate supervision of the instructor. Moreover, the slits at the ends of the two collimators form a very crucial part of the instrument, much more so than in the ordinary spectrometer. Consequently, whenever it is necessary to adjust them, do so with extreme caution.

The following order of operations is advised in making the adjustments:

1. The prism having been removed, adjust the instrument as described for the spectrometer (Appendix II, pp. 207–211). In this case there will, of course, be two collimators to adjust instead of one. In this and the two succeeding adjustments maintain the Nicol N_2 at its 90-degree position, that of complete transmission.

2. Replace the prism, and note that it is at such a height that the optical axis of the instrument cuts it at the edge of the silver strip. The prism should be placed with reference to the telescope and collimators approximately as illustrated in Fig. 95. Adjust the prism as described for the spectrometer on pages 77–78. At the conclusion of these adjustments set the prism at minimum deviation for the yellow-green portion of the spectrum, the light coming from collimator C_2. Clamp C_2 and the prism table.

3. The prism should now be calibrated by determining its dispersion curve in the way described for the prism spectroscope, using the collimator C_2 (see page 81). In this case leave the prism in the position in which it has been clamped. *Do not* set it at minimum deviation for each line as directed on page 81. It is not necessary to calculate Cauchy's A and B. Merely plot the dispersion curve for later use in translating readings from the divided circle into wavelengths.

4. Verify the adjustment of the Nicols. This should be such that extinction occurs when the polarizer N_2 is set at zero. This adjustment must be made with a very wide slit and with a bright source of light. The adjustment once made, set N_2 at 90 degrees to facilitate succeeding adjustments.

5. Precision in the use of this instrument often depends upon equality of width of the two collimator slits. The width of these slits is read by means of the indexes of the slow-motion screws controlling them. To make them of equal width proceed as follows: Set a line-spectrum source, such as a mercury arc, in front of one of the collimators. Close the slit by means of its slow-motion screw until the spectral lines *just* disappear. Read and record the index position. Open the slit to the width required, making the estimate by observing the spectrum through the telescope. Note the new index reading. Repeat with the other collimator slit, except that instead of judging the proper width of the slit in the foregoing manner, set the index so that the difference between the zero reading and the final setting is the same as in the case of the first slit.

6. Place identical line-spectrum sources in front of *each* slit. Two identical spectra will be seen, one displaced laterally with respect to the other. Rotate collimator C_1 until these two spectra become superposed; then clamp C_1.

7. *a.* If two continuous spectra are to be compared, the collimator slits must be of equal width for equal purity and accurate color matches.

b. If a bright-line spectrum is to be compared with a continuous spectrum, as narrow a slit as possible must be used on C_1. Hence some very bright source of light, such as a concentrated-filament automobile headlight, should be placed close to this slit.

c. If two bright-line spectra are to be compared, the width of the collimator slits need be only less than the minimum distance between the lines studied and other lines of appreciable intensity in their vicinity. The purity in this case is independent of slit width, provided only one spectral line is included.

8. The preceding adjustments are all made by looking through the eyepiece of the telescope. The eyepiece slit, which should be located in the principal focal plane of the telescope objective, is now narrowed until it exposes only the line or region in question, and left without further alteration during subsequent measurements.

9. The eyepiece is now removed. A rectangular diaphragm is placed over the telescope lens to limit the field of view to a simple rectangular aperture, one half of which is illuminated by light from the comparison collimator C_2, and the other half by light from the standard collimator C_1. Settings made by rotating the Nicols so that the two fields of view are of equal intensity must be made with the eye as close to the eye slit as possible. To insure the eye's remaining near the optical axis of the telescope, the eye slit may be shortened by a horizontal slit behind it, so that vision is obtained essentially through a simple rectangular aperture.

Measurements. 1. *Continuous spectra.* Set the sources to be compared before the two collimators, placing before C_1 that which is used as a standard. Arrange their distances from the collimators so that as nearly as practicable the intensities of the two are the same at some chosen point of the spectrum when N_2 is set at 90 degrees. Secure a photometric match at as many points of the spectrum as is deemed necessary. In general at least ten such points will be advisable. For each point make at least five independent settings of N_2, making these settings alternately from opposite sides. For each point, record the setting of the divided circle of the spectrometer and the mean of the five settings of N_2. Plot as ordinates the relative intensities (taken from the $\sin^2\theta$ table of Appendix III) against wave-lengths (taken from the dispersion curve) as abscissæ. The resulting curve will show the intensity of the source under investigation in terms of that used as the standard throughout the visible spectrum.

2. *Absorption spectra.* In the same way as in the preceding paragraph, determine the wave-length-intensity curve of light which has passed through a solution or color filter prescribed by the instructor. It will be found convenient to arrange a pair of mirrors so that light from the same source enters both collimators, that entering C_2 having passed through the solution under investigation. Repeat for other such solutions or filters. In this case both sources should be the same, the solution under investigation being placed between the collimator C_2 and the corresponding source.

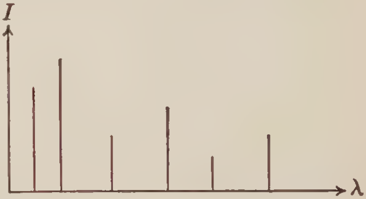

Fig. 96 A. Graphical representation of intensities in discontinuous spectrum

3. *Discontinuous spectra.* Observing the caution under 7, *b*, of the preceding page, deduce the wave-length-intensity relation for some discontinuous spectrum, assigned by the instructor, in terms of the standard source. In this case the relation will not be represented by a continuous curve, as in the preceding cases. It may be represented as illustrated in Fig. 96 A.

APPENDIX I

ILLUMINATION

There are several phases to the problem of illumination for a laboratory course in light. Some experiments require brilliant illumination and others almost total darkness, while still others require both. Added to the necessity of effecting a satisfactory adjustment between these conflicting requirements is the ever-present problem of monochromatic illumination.

Most physics laboratories are provided with a properly designed room for a course of this nature. For the few that are not, the following suggestions may not be amiss: The laboratory room must be so designed that it is possible to shut out all external light, whether from windows, doors, or transoms. Ordinary window shades are totally inadequate and must be replaced or supplemented by some form of light-tight screens. Painting walls and woodwork black ruins the room for uses other than that of a light laboratory, and in general will not be necessary. A liberal supply of light wooden or sheet-metal screens, both fixed and movable, will be found exceedingly useful. It is probably unnecessary to remark that a room closed as tightly as is necessary for experimental work in light should be provided with special ventilating facilities.

For nearly every experiment outlined in this text two plug outlets, for electric-light extensions, and a gas outlet are indispensable. This set of three outlets should be permanently installed at each table and at intervals of every four or five feet along wall benches. Neither compressed-air outlets nor storage-battery connections are necessary, though it will be found convenient to have them available at one or more points in the room. Frosted electric lamps will be found more satisfactory than clear globes. Wire lamp guards provided with opaque shields covering one half of their area are advisable. These guards should be locked to the sockets in such a way that they cannot be removed except by the use of a key. All extension sockets should receive some identifying mark; otherwise a necessity will rather frequently arise for replenishing the supply. A properly shielded right-angled carbon arc mounted on a base, together with its rheostat and switch, will be found convenient for occasional use where an intense white-light source is needed.

The type of monochromatic illuminant which is best adapted to individual experiments has in general been specified in the foregoing laboratory instructions. The mercury arc, the sodium flame, and the helium tube are the sources most commonly used. Light from the mercury arc, when passed through an appropriate color filter, such as the green "mercury monochromat" Wratten filter, forms an excellent monochromatic source of sufficient intensity for many purposes. Commercial mercury-in-glass arc lamps (either alternating or direct current) form a dependable source for such illumination. They are rather bulky, but they may sometimes be arranged with a system of mirrors to supply illumination for several experiments simultaneously.

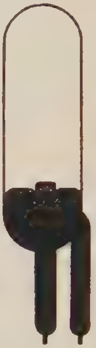
FIG. 97. Pyrex mercury arc lamp

The author has found the small mercury-in-pyrex arc lamp illustrated in Fig. 97 to be very convenient. It is not sold by any firm in America, but may be readily made by a glass-blower. After having been filled with mercury it may be exhausted by an ordinary oil pump and sealed off. In use it is set on a base similar to that illustrated in Fig. 98. The two wells in this base are to be filled with mercury. It should be connected to a direct current of 110 volts, through a resistance of about 30 ohms, with the positive terminal connected to the center electrode. Until the lamp becomes heated it requires shaking at intervals, to reëstablish the arc. This may be obviated by preheating it to about 200° C. in an electric furnace. While in use it should be shaken vigorously, about every five minutes, to replenish the mercury in the center tube, the level of which gradually lowers from evaporation as the current flows.

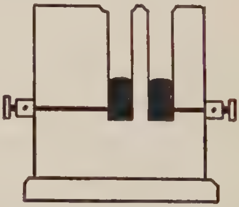
FIG. 98. Base for mercury arc lamp

For some experiments an ordinary sodium source is preferable to any other. It would perhaps be appropriate to point out that for experiments of the type of the Fresnel mirrors and prism an exceptionally intense source is required, and that the oxy-gas flame, as described on page 42, fills the requirements quite satisfactorily. A pressure-reducing valve is not necessary; in its place a needle valve may be used. In either case it is well to educate the students in the use of the control valves of the oxygen tank. Otherwise much oxygen will be wasted, and ultimately the reducing valve or the needle valve, as the case may be, is certain to be ruined. A prominent tag, permanently attached to the valve and bearing the following instructions, duly capitalized and underscored, will be found effective:

TO START

Open top valve wide.
Adjust flow with side valve.

TO STOP

Turn top valve clear down hard.
Exert no force on side valve.

For the calibration of prism spectroscopes some source which gives a bright-line spectrum containing twenty or thirty lines rather uniformly distributed along the visible spectrum is required. A combination of the mercury arc and a helium discharge tube will be found to fill this requirement for many purposes. The more prominent lines in each case are as follows:

Mercury		Helium	
Wave-length	Color	Wave-length	Color
5790	yellow	7065	red
5770	yellow	6678	red
5461	green	5875	yellow
4916	blue-green	5015	green
4359	blue	4922	blue-green
4078	violet	4713	blue
4046	violet	4471	violet

Another convenient source is an amalgam made up of mercury, cadmium, lead, antimony, and zinc, the amalgam being substituted for the mercury in the lamp illustrated on page 198. The following lines will be seen in this arc:

Wave-length	Color	Wave-length	Color
6438	red	4949	blue
6325	orange	4916	blue
6194	orange	4811	blue
6130	orange	4800	blue
6104	orange	4722	blue
5790	yellow	4678	blue
5770	yellow	4663	blue
5609	yellow	4418	violet
5490	green	4359	violet
5461	green	4339	violet
5339	green	4168	violet
5086	green	4078	violet

With this lamp a modification of the design illustrated in Fig. 97 will be advisable, because of the tendency of the amalgam to form a film which short-circuits the lamp. The difficulty may be avoided by sealing into the top of the lamp a third wire, terminating in a large iron electrode. After the lamp has been heated, the circuit should be from the lower side electrode to this upper one.

APPENDIX II

LABORATORY MANIPULATIONS

I. The Silvering of Glass Surfaces

Cleaning glass surfaces for silvering. The most important part of the silvering process is the proper cleaning, both of the surface to be silvered and of all containers and utensils. Proper attention to this item will eliminate most of the difficulties commonly encountered. The following process of cleaning is recommended:

1. Remove grease and wax, if they are present in visible quantities, with alcohol or carbon disulphide.

2. Pour over the plate a 10 per cent solution of caustic potash. While the plate is under this solution, rub it well with a tuft of clean cotton tied to the end of a clean glass rod. Rinse thoroughly.

3. Place the plate in the beaker in which the silvering is to be performed, and from which it should not be removed until the silvering is complete. Pour over it concentrated nitric acid, and rub as before, with another tuft of cotton.

4. Wash the plate thoroughly in running water, raising it with a glass rod so that all traces of acid underneath it are removed. At the same time rub vigorously with a cotton tuft on the end of a glass rod. Rinse the plate in several changes of distilled water, raising it as before. Finally, leave it in such a depth of distilled water as will just cover it, until it is to be silvered.

During the cleaning process every care should be taken not to touch the glass plate or the inside of the container with the fingers. A microscopic trace of organic matter is sufficient to cause streaky silver films. The best test of cleanliness is to observe that as the plate is lifted from the water a uniform film clings to the surface. Unless this is the case, the cleaning is insufficient. Scrupulous cleanliness must also be observed in preparing the silver solutions and their containers.

Silvering glass surfaces. Two general methods of silvering are in common use: chemical deposition and cathode deposition. The former has the advantage of not requiring expensive apparatus, and the latter the advantage of greater uniformity and hardness of the film. Moreover, chemical deposit becomes more difficult as the surfaces age. New surfaces take such a deposit easily; but after silvering them a number of times, it is difficult to produce a uniform coat in this way. The cathode-deposit method possesses a distinct advantage here, for an old plate receives as good a coat as a new one, except, of course, for the effects of actual abrasions.

Brashear's process. Of the chemical methods in common use, the one known as *Brashear's process* is as a rule the most satisfactory. Three solutions are required. One, the reducing-solution, should be prepared at least a week before it is to be used. Its quality improves with age, apparently without limit. It may be made in quantity and kept in stock with advantage. We shall call this Solution C. Solutions A and B will keep indefinitely, but age seems not to add to their quality. These solutions are made up as follows:

Solution A:	
Water	500 cc.
Silver nitrate	25 g.
Solution B:	
Water	250 cc.
Potassium hydroxide	25 g.
Solution C:	
Water	2000 cc.
Rock candy	180 g.
Nitric acid (C.P.)	8 cc.
Ethyl alcohol	350 cc.

To use, proceed as follows:

1. Dilute 20 cc. of Solution A with 100 cc. of water. Add a few drops of dilute aqua ammonia. The solution will turn to a dark-brown color. Continue to add ammonia little by little, shaking continuously until the brown precipitate *just* disappears.

2. Add 10 cc. of Solution B, which will again produce a precipitate. This precipitate is to be nearly, *but not entirely*, redissolved by the further addition of ammonia, a few drops being sufficient this

time. If too much ammonia is accidentally added, so that the solution becomes completely clear, a drop or two from the stock supply of Solution A will bring back traces of the precipitate. The final mixture *must* show the brownish precipitate in sufficient amount to render it a light straw color.

3. The addition of 5 cc. of Solution C will cause precipitation of silver to begin almost immediately. This should be the final step in the preparations for silvering, everything else being ready before this is carried out.

Depositing an opaque film. In case an opaque film is desired, throw off the water which covers the plate to be silvered, and pour the final mixture over the plate in such quantity as to cover it with not less than 1 cm. of liquid. The solution turns to a black color, which in a few minutes will turn brown and later gray. At this time, which will be in from four to ten minutes, depending upon conditions, the precipitate becomes flocculent, the individual grains being easily visible. The first coat is now completed. If the mirror remains in the solution too long, it will take on a grayish color which cannot be removed even by polishing.

Throw off the old mixture, rinse with distilled water, and pour in fresh solutions to the same amount as before. When the second coat is completed, remove the mirror, immediately rinse it with distilled water, and set it on edge to dry.

The quality of the silver film will be improved if continually, during the deposition of both coats, the surface is lightly brushed (*not rubbed*) with a small pad of cotton on the end of a glass rod. This brushing-process should be started just after the coating first becomes visible on the mirror. Needless to state, the cotton must be perfectly clean, the portion of it which comes in contact with the mirror not having been touched by the hand.

A uniform temperature of the bath and plate of about 20° C. is essential. Since fulminating silver is liable to be produced by the action of ammonia on silver oxide, all used mixtures should be thrown away as soon as the silvering operation is completed. The silvered plate should remain undisturbed for the greater part of a day, in order that it may become thoroughly dry. It may then be polished, if necessary, with a small pad of chamois stuffed with cotton, on which is spread a little optical rouge.

Depositing half-films. In case a half-coat is to be deposited, the silvering-mixture should be poured directly into the distilled water which covers the previously cleaned plates. The plates should be carefully watched, and should be removed when it is estimated that a coating of sufficient thickness has been deposited. This estimation requires experience. No definite time can be set, since the rate of deposition varies with concentration and with temperature. Fifteen or twenty seconds after individual grains of silver are first seen to float on the surface of the mixture will be about the proper time to remove the mirror. This will be in from one to two minutes after the deposition starts. Upon removal from the solution the plates should be immediately rinsed with distilled water and set on edge to dry.

For the half-silvered plates of the Michelson interferometer such a thickness should be deposited that a little more than half of the light incident at 45 degrees will be reflected. In this case it is well to silver all four faces of the two plane-parallel plates and to select for use that which is of proper and most uniform thickness. For Fabry-Perot plates the thickness ought to be somewhat greater. A simple method of testing is to look at two similar electric lights, one seen through the film and the other seen reflected by it.

Polishing half-films. Polishing these half-films is a decidedly delicate matter. For the Michelson interferometer the necessity for it is scarcely great enough to justify the trouble of acquiring the necessary skill. With the parallel-plate interferometer, however, the contrary is true. Success in the use of this instrument invariably demands a polish on the plates which is seldom or never present as the plates come from the silvering-process.

An indispensable device for this polishing process is a powder puff of extremely light feather down, such as is commonly found among babies' toilet articles. The half-film when brushed lightly (not rubbed) with this puff, into which has been dusted some very fine optical rouge, assumes an excellent polish. The process becomes rather arduous, however, if carried out to a sufficient degree to become really effective. The labor may be considerably lightened by using the puff as a buffing-wheel, its handle being placed in the chuck of a lathe.

The material of which this type of puff is made may be purchased by the yard at some dry-goods establishments under the name of "swan's-down." With this material at hand, buffing-wheels for the purpose may be readily made of such dimensions as will doubtless be more convenient than is the puff itself.

Cathode-deposit process. This process utilizes the fact that when an electric discharge occurs in a partial vacuum, the cathode stream carries with it minute particles, possibly atoms, of the material of the cathode. These are projected in straight lines, and are deposited on whatever obstacles are in their path. Some metals are deposited more readily than others. It is fortunate that silver and, to a slightly less degree, gold, platinum, and a few other metals lend themselves readily to this method of deposition. Fig. 99 shows the arrangement of apparatus for performing this process. Detailed description of the apparatus is not necessary. The process is simple though it requires some experience. The following table explains the notation of the figure:

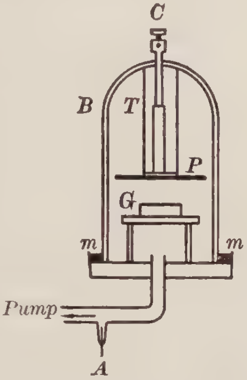

FIG. 99. Apparatus for silvering by means of cathode deposit

B, bell-jar on a glass base.
P, plate of the metal to be deposited.
G, glass plate prepared to receive the deposit, from 2 to 4 cm. from the metal plate.
C, cathode, which should have an area greater than that of the surface to be silvered.
A, anode.
m, mercury seal.
T, glass tube inclosing the supporting-rod of the metal plate.

An induction coil may be used, but alternating current through a transformer will be found more satisfactory. A transformer of the type used in small radio sending-stations suits the purpose admirably. Using a transformer, deposit will occur from both electrodes. As a rule this is not objectionable. It is, however, readily eliminated by using an anode of aluminum, which does not readily "sputter," and by introducing one or more bends in the tubing connecting the anode with the bell-jar, as well as by removing the anode to a considerable distance (two feet or even more).

The amount of current advisable cannot be specified. Only experience will give reliable information. In general, use as heavy a current as possible without endangering the seals or mirror by sudden and excessive heat. It is convenient to use a milliammeter to indicate current, though this is not a necessity.

A fairly high vacuum is necessary. Any pump which is incapable of producing an X-ray vacuum will be found inadequate. One of the more recent types of oil pump, if in good condition, will be found quite satisfactory. A liquid air trap to eliminate water and organic vapors, as well as to exclude mercury vapor if a diffusion pump is used, is convenient but not indispensable. A side tube of phosphorous pentoxide is advisable if liquid air is not used. The proper degree of vacuum is one of the most important items of manipulation, but it is not easy to specify. It should be of the order of a few hundredths of a millimeter of mercury. The best indication is the appearance of the bell-jar. This should be almost completely filled with a soft glow, the Crookes dark space being from one to two centimeters from the surface that is being coated.

The plate to be coated should have been previously prepared as described on page 201, though the requirements of cleanliness are not as exacting as in the case of chemical deposition. The inside of the bell-jar and all exposed surfaces should be cleaned thoroughly with a caustic-potash solution followed by nitric acid. If gold or platinum has been previously deposited, the cleaning must, of course, be performed with aqua regia. After thorough rinsing in distilled water, all such surfaces, including the plate to be silvered, should be dried by vigorous rubbing with clean cotton. After silvering, the apparatus should be allowed to cool before air is admitted.

Silvering-cabinet. A special cabinet containing the supplies necessary in silvering will be found a great convenience. It will be advisable to locate this cabinet near a sink. Most physics laboratories are equipped with a chemical-preparations room, and such a room is the logical place for this cabinet. It will be well to provide a lock, since the exacting requirements of cleanliness render it unwise to allow other preparations to be made with this equipment. The following supplies should be provided. Those marked with an asterisk will not be necessary where the cathode-deposit method is used.

For the silvering-cabinet:

500 cc. potassium hydroxide (10 per cent solution).
500 cc. nitric acid (C.P.).
500 cc. ethyl alcohol.
*One 600-cc. bottle marked "Solution A."
*One 300-cc. bottle marked "Solution B."
*One 3000-cc. bottle marked "Solution C."
One 500-cc. graduate.
One nest of evaporating-dishes.
Six pieces of glass rod 6 inches long.
Roll of cotton.
*Glass funnel.
*Ring stand.
Filter paper.

Available, but not necessarily in the cabinet:

Rough balance.
Distilled water.
*Silver nitrate.
Potassium hydroxide.
*Rock candy.
*Ammonium hydroxide.

II. The Adjustment of the Spectrometer

Aside from the spectrometer to be adjusted, two simple auxiliaries will be found convenient; namely, an optically plane-parallel glass plate mounted on a support with leveling-screws, and a source of diffuse white light, such as would be produced by an incandescent electric lamp with frosted globe. The plate should preferably, though not necessarily, be silvered on both sides.

The spectrometer. The following description of the spectrometer, and also the instructions for adjusting, have been adapted with a few changes from D. C. Miller's "Laboratory Physics."

The spectrometer is essentially an instrument for the measurement of prism angles and of deviation of light rays. It has for its component parts a horizontal table with a divided circle, a telescope, and a collimator. The telescope and table are capable of independent, measurable rotations about the principal vertical axis of the instrument, which passes through the center of the circle perpendicular to its plane; the collimator is commonly attached to the base of the instrument, and can be rotated about this axis only with the instrument as a whole. For most purposes it is required that the axes of the telescope and collimator should intersect the axis through the center of the circle and should always be in the same plane perpendicular to it. In order that the spectrometer may be made to satisfy these conditions and that it may be used for a variety of purposes, it is usually provided with many movements, adjusting-devices, and graduated scales which need not be described. Before the spectrometer is used for any work of precision, the accuracy of its adjustments should be verified. This may be done in various ways, but the following method is recommended as being concise and accurate. It is important that the various operations be performed in the order described.

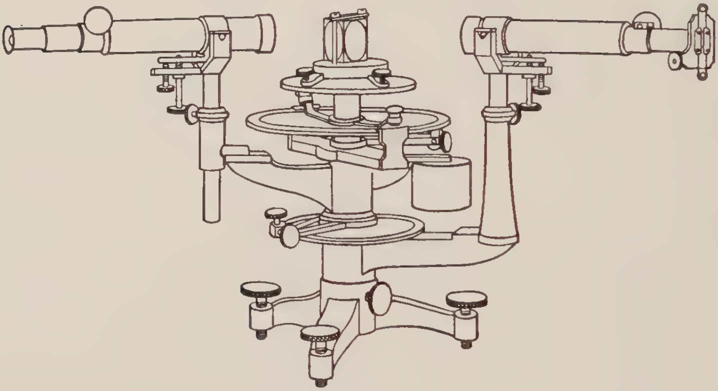

FIG. 100. Spectrometer

Adjustments. *Preliminary.* Level the table of the instrument approximately, by means of the screws in the base. Make the axes of the telescope and collimator approximately horizontal.

Set the table (or the telescope and collimator) at such a height that the prism or other apparatus to be used will be centrally located in the line of sight. If the instrument is so designed that the telescope and collimator can be rotated about any axis other than that of the instrument, turn them so that their axes shall intersect this main axis. This is done most easily by viewing through the telescope and collimator tubes, in turn, the image of a pencil held upright at the center of the table, the slit and eyepiece having been previously removed.

Focusing telescope for parallel light. If possible remove the eyepiece and cross-hair cell in their common mounting. Take the mounting to a window and, directing it toward the sky, adjust the eyepiece until the cross hairs are in perfect focus. Replace, and direct the telescope toward some distant object. Focus carefully until no parallax between the image and the cross hairs is observable. If the eyepiece is not of the collimating (Gauss) type, this is the best adjustment that can be made for parallel light. If a Gauss eyepiece is available, however, this adjustment can be greatly improved. The Gauss eyepiece carries within it a thin plate of glass inclined at 45 degrees in order to direct along the axis of the telescope the light received through an aperture in the side of the eyepiece (see Fig. 101). This eyepiece may be used in the way described below, to improve the adjustment for parallel light.

Set the glass plate on the spectrometer table, setting it vertical by inspection. Place a diffuse light in the position indicated in the figure. Looking through the telescope, slowly turn the spectrometer table one way and the other until (when the mirror is normal to the axis of the telescope) a circular patch of light appears. If this does not occur, turn the leveling-screw of the telescope a half-turn either way and resume the search. When the reflected beam is found, make such adjustments as are necessary to cause the entire field of view to be illuminated.

Observe the field carefully in search of a second image of the cross hairs. If it is not seen, slowly focus the telescope by means of the rack and pinion. When it is found, readjust until both images of the cross hairs appear distinct, and until motion of the eye ceases to disclose any parallax. When this is accomplished, the telescope has been focused for parallel light, and its focus should not again be disturbed throughout the entire experiment.

Making axis of telescope perpendicular to axis of table. For this and the succeeding adjustments the Gauss eyepiece is indispensable. Rotate the spectrometer table and alter the leveling-screw of the telescope to bring the reflected image of the cross hairs into coincidence with the direct image. Rotate the spectrometer table (not merely the mirror stand) until the second face of the mirror is turned toward the telescope, and find the image of the cross hairs reflected from this face. Cause this image to coincide with the direct image by the following manipulations: first, rotate the table until the reflected image is vertically above or below the direct image; second, turn the leveling-screw of the telescope far enough to move the reflected image only *halfway* toward the direct one; third, alter the leveling-screws on the mirror stand until exact coincidence is obtained. If the adjustments have been accurately made, coincidence should now be obtained from the first face when it is again turned to the telescope. If upon trial it is not exact, correct half the error by altering the inclination of the telescope, and correct the other half by tilting the mirror, in the same way as before. Continue in this manner until the reflected image from each face coincides with the direct image. Turn the plate (not the prism table) through 90 degrees and repeat. The axis of the telescope should now be perpendicular to the axis of the table, and the surfaces of the mirror are parallel to this axis.

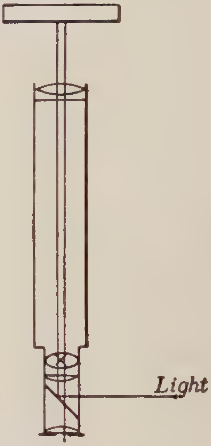

Fig. 101. Telescope with a collimating eyepiece

Making axis of telescope perpendicular to its axis of rotation. The axis of rotation of the telescope should be coincident with the axis of the table. As the spectrometer is ordinarily constructed, the relation of these two axes cannot be altered. If the two axes are not parallel, the axis of the telescope can be made perpendicular to either, but *not*, of course, to *both*. Evidently it should be adjusted with respect to that axis about which rotation is to take place in the contemplated measurements.

The preceding adjustment having been completed, the accuracy of construction of the instrument in this respect may be quickly tested. For the mirror is parallel to the axis of the table, and the telescope is perpendicular to the mirror. If now the telescope is turned 180 degrees about the central axis, it should still be perpendicular to the mirror if the axis on which it turns is parallel to that of the table.

If it is desired to set the optical axis of the telescope perpendicular to its axis of rotation, the method is similar to that of the preceding adjustment, except that the telescope is rotated from one side of the mirror to the other instead of the mirror's being turned from side to side by the rotation of the table. When the spectrometer is used for measurements on a prism, it is usually more important that the axis of the telescope be perpendicular to the axis of rotation than that it be perpendicular to the axis of the table.

Rendering surface of table perpendicular to its axis. The telescope and mirror being adjusted for coincidence of the direct and reflected images of the cross hairs, carefully lift the mirror above the table, rotate the table through 180 degrees, and replace the mirror. If this operation is found to have disturbed the coincidence in a vertical direction, take up half of the difference with the leveling-screws of the mirror stand and the other half with those of the table. Turn the mirror through 90° and repeat. This adjustment is usually unimportant.

Adjusting collimator. Remove the mirror, and sight the telescope through the collimator upon the slit; cause the latter to be seen distinctly, without parallax with respect to the cross hairs, by sliding the draw tube of the slit. Do not alter the focus of the telescope.

Turn the slit horizontal, and level the collimator by bringing the slit to the cross hairs. Replace the slit in a vertical position and do not again alter the adjustments.

III. The Use of the Vernier Scale

A vernier is an auxiliary sliding scale which permits the observer to increase the accuracy of his estimation of fractional portions of the smallest division of the main scale. Usually verniers are so ruled that $n-1$ of the main-scale divisions correspond to n divisions on the vernier, and the smallest indication thereby given is $\frac{1}{n}$ of a division. For example, in Fig. 102 nine divisions of the main scale are spanned by ten on the vernier, and the smallest indication is in

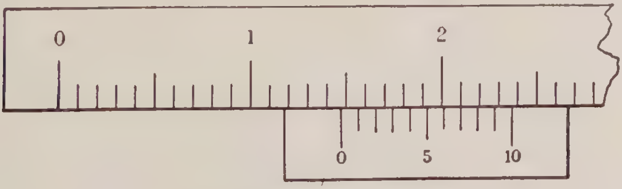

Fig. 102. Scale with vernier

tenths of a division of the main scale. The zero of the vernier is to be taken as the main-scale index. The reading in Fig. 102 is 1.47.

The circular scale of the spectrometer is usually read with the aid of a vernier. A common form is illustrated in Fig. 102 A, which shows a circular scale in which fifty-nine divisions of the main scale are spanned by sixty of the vernier. The vernier therefore gives an indication of one sixtieth of the smallest main-scale division. Since these divisions are spaced one sixth of a degree, or ten minutes, apart, the vernier enables the observer to read the instrument to ten seconds. Accordingly the vernier is graduated from zero to ten. The reading illustrated is 158° 47′ 10″.

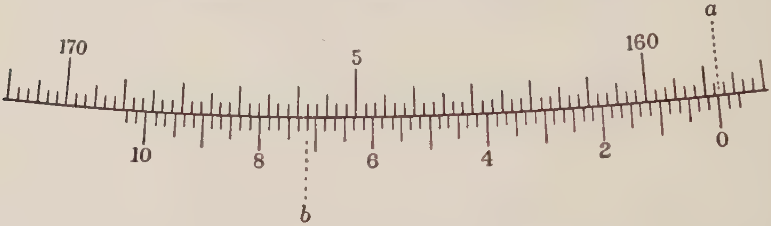

Fig. 102 A. Vernier for divided-circle reading to 10″

IV. The Use of Micrometer Calipers

A common type of micrometer caliper is illustrated in Fig. 103. The method of use is as follows:

1. Determine the *zero reading*; that is, the reading of the micrometer head when the jaws are in contact. In general this will not be zero.

2. Determine the reading when the object to be measured is between the jaws. In the usual type of metric instrument graduations on the horizontal scale are in millimeters, and on the vertical in hundredths of a millimeter. As a rule the pitch of the screw is a half-millimeter. Take care, therefore, to note whether the reading is, for example, $4.\underline{0}72$ or $4.\underline{5}72$. The reading on the vertical scale is the same in both cases.

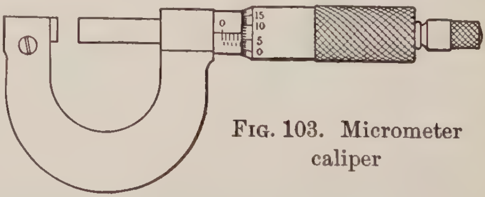

Fig. 103. Micrometer caliper

3. Apply to the reading the zero correction found in IV, 1, on the preceding page, with proper attention to sign.

In using the micrometer calipers take great care always to exercise the same pressure on the milled head. In the better types of instrument the head is equipped with a ratchet designed to slip as soon as a certain pressure is exceeded. In this case the ratchet should be caused to slip always the same number of notches. Always take a mean of not less than five settings.

V. The Use of the Spherometer

A common type of spherometer is illustrated in Fig. 104. It is a somewhat more precise instrument for the measurement of short lengths than is the micrometer caliper. It receives its name from the fact that it may be used to determine the radius of curvature of a spherical surface.

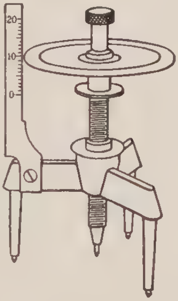

Fig. 104. Simple spherometer

Handle the spherometer carefully, since it is extremely easy to damage it in minor respects. Contact of the screw point with the surface upon which measurements are being made may be detected by the tactile sense of the experimenter or by observation. In the former case, the instrument being held down firmly with one hand, contact is detected by the difference in the resistance which the screw presents to rotation. In the latter case, notice of contact is given by the instrument's "hobbling," or rocking. The screw is then carefully turned back until this rocking just ceases.

Always take the mean of not less than ten readings, coming each time up to the final position from the same side. In using the instrument to measure the thickness of some object, proceed as follows:

1. Determine the zero reading of the instrument. For this purpose there should be provided a square of plate glass. In Experiment II, *A*, the plane side of the lens may be used. Turn the micrometer screw and disc until the zero mark of the disc is next to the vertical fiducial edge. Observe whether the disc is exactly opposite one of the marks on this edge. If not, hold the screw firmly, and rotate the disc without turning the screw (a friction device permits this) until the

instrument is in adjustment in this respect. Usually the pitch of the screw is a half-millimeter. Take care, therefore, to observe whether a reading is, for example, $27.\underline{3}532$ or $27.\underline{8}532$. The reading on the circular scale will be the same in both cases.

2. Take the reading when the object whose thickness is to be measured is beneath the point of the screw.

3. The difference between the two readings gives the required thickness, a. If the spherometer is used to measure a radius of curvature, the following additional steps will be necessary:

4. By indentation secure the impression of the three leg points of the spherometer on a piece of paper. Measure the distance between each pair and take the mean.

5. To find the radius R, substitute the value of the difference a of the two spherometer readings, and of the mean distance d between the legs of the instrument, in the relation

$$R = \frac{d^2}{6a} + \frac{a}{2}.$$

VI. The Use of the Micrometer Microscope

Fig. 105 shows the appearance of the field of view of a common type of micrometer microscope. At the head of the micrometer screw is usually a disc with a scale divided into one hundred parts for reading fractions of a turn. The saw-toothed index is intended to facilitate the counting of whole turns without removing the eye from the eyepiece.

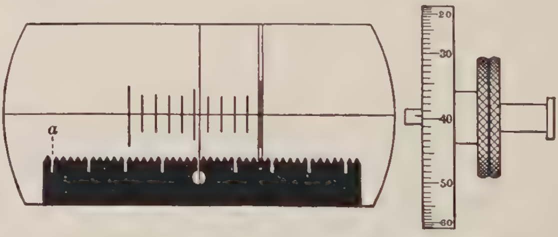

Fig. 105. Field of view of micrometer microscope

The disc is usually held in position by a friction washer to permit setting it to a proper position. When the moving cross hair coincides with one of the notches of the saw-toothed index, this disc should be at the zero position.

In focusing, first bring the cross hairs into good focus by sliding the eyepiece only; then bring the object into focus by moving the whole microscope tube. It is essential to *eliminate parallax*; that is, relative motion of the cross hairs and the image when the eye is moved sideways.

In making a setting the cross hair should always be brought up from the same side, to eliminate the effect of lost motion in the screw. Every reading should be the mean of at least ten settings of the screw.

The measurement of a length by the micrometer microscope consists of three stages: (1) observation of the required length in terms of the number of turns of the micrometer screw; (2) calibration of the micrometer by comparison with a standard scale; (3) division of the length as measured in terms of the turns of the screw by the calibration constant, expressed in turns per millimeter. Step (2) should never be omitted. Since the constant of the instrument changes with every change of focal length, it is not safe to accept any previous determination. Moreover, care should be taken that no change of focal length occurs in the process of a series of measurements.

VII. The Use of the Analytical Balance

Weights. Check your box of weights immediately upon receiving it.

The weights provided are sufficient for all the weighing you will do. *Never borrow from a neighbor's supply.*

No rider is provided. It is unnecessary.

Handle the weights only with forceps. If you find it necessary to lay them down, provide a sheet of clean paper to receive them.

Arrange the weights in the box, and check them at the conclusion of the weighing.

Balance. You will handle the balance by two controls, known as the *beam arrest* and *pan arrest.* Familiarize yourself with their action before starting to weigh.

Never touch either the pan or its contents unless the beam has been raised and clamped by the beam arrest.

Clamp the beam arrest only at the instant when the swinging pointer is passing through its zero point (see paragraph 2, below).

When taking readings for a rest point. close the front of the balance to eliminate air currents.

Weighing by interpolation. 1. Clean the pans of any dust or other accumulation. Lower the beam arrest, and gently manipulate the pan arrest until the pointer is caused to swing from four to eight divisions on each side of the mid-point of the scale, the pans being empty.

2. Record the extreme positions of five successive swings of the needle, and average. Call this average the zero point a.

3. Place the object to be weighed on the *left* pan, and place weights on the *right* pan in such an amount that the approximate mid-point of the swing of the pointer is two or three divisions to the *right* of the zero point. Determine the rest point b of the needle as in the preceding paragraph.

4. Add two milligrams and determine the new rest point c. This rest point should lie to the *left* of the zero point. If this is not the case, discard b, call the last rest point b, and repeat for c.

5. To find the true weight, add to the weight corresponding to b an amount given by the expression

$$\frac{b-a}{b-c} \times .002 \text{ g.}$$

Discard all figures beyond the fourth decimal.

6. In case of double weighing, for the second weighing change these instructions as follows:

In paragraph 3 change "right" to "left," and "left" to "right."

In paragraph 4 change "left" to "right."

In paragraph 5 change $\frac{b-a}{b-c}$ to $\frac{a-b}{c-b}$.

VIII. ADJUSTMENT AND USE OF THE TELESCOPE AND THE SCALE FOR ANGLE MEASUREMENT

A description of the apparatus used is unnecessary. Directions may be most profitably arranged in numerical order as follows:

1. Point the telescope toward the light, and focus the eyepiece on the cross hairs.

2. Place the eye near the position to be occupied when using the telescope, and find the image of an electric light or of some well-illuminated object in the mirror. Move the eye and the lamp simultaneously until the lamp is directly below the eye. Remove the lamp and set the scale in its place. Henceforth use the lamp to illuminate the scale, if such illumination is necessary.

3. Place the telescope in front of the eye and, looking *over* it, point it at the mirror.

4. Adjust the height of the scale until it becomes visible, still looking over the telescope.

5. Looking *through* the telescope, adjust the focusing-screw until the *mirror* appears.

6. Shorten the telescope by the focusing-screw until the *scale* appears.

7. Eliminate the parallax by careful focusing.

8. Turn the mirror until the portion of the scale seen is that near the telescope.

9. To measure the angle through which is turned the object to which the mirror is attached, (1) observe the change d in the scale-reading when the mirror rotates; (2) measure the distance D from the mirror to the scale; (3) substitute the measured values of d and D in the relation $\alpha = \frac{d}{2D}$, to find the angle α through which the mirror has turned. This relation, while not exact, is sufficiently accurate for small angles.

IX. THE BABINET COMPENSATOR

The construction of the Babinet compensator was described in Chapter XI (p. 153). Its operation and adjustment will be more fully discussed here. This discussion is taken from Mann's "Manual of Advanced Optics."

Suppose light, polarized in a plane which is normal to the plane defined by the two optic axes of the wedges, and which bisects the angle between those axes, to fall perpendicularly upon the wedges as indicated by the line NN' (Fig. 106). On entering the wedge A, the vibration will be separated into two equal components, one parallel to the edge of the wedge, and the other perpendicular to it. These components will travel with different velocities within the wedge, and will therefore emerge with a difference of phase, the magnitude of which will depend upon the color of the light used and upon the thickness of the wedge at the place where the light has passed through. On entering the wedge B, the light again separates into two component vibrations which travel with different velocities; but because the optic axis of this wedge is perpendicular to that of the other wedge, the component vibration which traveled slower in the wedge *A* will travel faster in the wedge *B*, and vice versa. It is evident, then, that along that line parallel to the wedge edges, where the light has traveled through equal thicknesses of the two wedges, it will emerge in exactly the same condition in which it entered, while along other lines its condition will depend on the difference of the thicknesses through which it has passed.

If such a pair of wedges is brought between crossed Nicols, and so oriented that the optic axes make angles of 45 degrees with the planes of transmission of the Nicols, the line along which the thickness of the two wedges is the same will appear as a dark band, because along it the condition of polarization of the incident light is not altered by the compensator. If the combination is illuminated with white light, colored bands will be seen on both sides of this dark band. If the combination is illuminated with monochromatic light, a series of equidistant dark bands will appear, the distance between the successive bands showing how far it is necessary to move along the wedge in order to introduce a difference of phase of a whole wave-length of the particular light used.

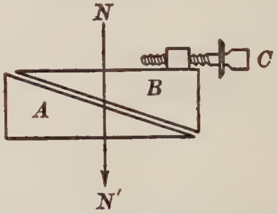

FIG. 106. Cross section of Babinet compensator

If the wedge *B* is moved by the screw *C*, the bands will move, and the constant of the instrument is simply the number of turns which must be given the screw *C* in order to make the dark bands move up by one; that is, the number of turns which corresponds to a difference of phase of a whole wave. By turning the screw *C* through a fraction of the number of turns which correspond to a difference of phase of a whole wave, a difference of phase of that same fraction of a wave is introduced along the line formerly occupied by the central dark band. Thus the Babinet compensator allows us to introduce any difference of phase which may be desired. It may also be used to measure a difference which has been introduced by other means; for any such difference of phase will shift the central band, and may be measured by counting the number of turns of the screw *C* which are needed to cause the bands to return to their original position. This number, divided by the constant of the instrument, will be the required difference of phase. It is because of its use in this way, to compensate a phase difference which has already been introduced, that the instrument has received its name.

In use, the compensator is mounted in the focal plane of the eyepiece of the instrument with which it is to be used. A cross hair should be fastened to it, to mark the position of the central black fringe, and the eyepiece should be focused on this cross hair. The constant of the Babinet must then be determined for the wave-length to be used, in the way described in the foregoing paragraph.

One way of using the compensator in connection with Experiment XIII is to set it so that it will introduce a difference in phase of a quarter wave, and to rotate it until the central black band returns to the zero position. The analyzer must then be rotated until the central black fringe is blackest. The directions of the optic axes of the compensator will show the directions of the principal axes of the ellipse, and the tangent of the angle between one of those axes and the plane of transmission of the analyzer is the ratio of the axes.

SIN^2 θ FOR SPECTROPHOTOMETER

°	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002
1	.0003	.0004	.0004	.0005	.0006	.0007	.0008	.0009	.0010	.0011
2	.0012	.0013	.0015	.0016	.0018	.0019	.0021	.0022	.0024	.0026
3	.0027	.0029	.0031	.0033	.0035	.0037	.0039	.0042	.0044	.0046
4	.0049	.0051	.0054	.0056	.0059	.0062	.0064	.0067	.0070	.0073
5	.0076	.0079	.0082	.0085	.0089	.0092	.0095	.0099	.0102	.0106
6	.0109	.0113	.0117	.0120	.0124	.0128	.0132	.0136	.0140	.0144
7	.0149	.0153	.0157	.0161	.0166	.0170	.0175	.0180	.0184	.0189
8	.0194	.0199	.0203	.0208	.0213	.0218	.0224	.0229	.0234	.0239
9	.0245	.0250	.0256	.0261	.0267	.0272	.0278	.0284	.0290	.0296
10	.0302	.0307	.0314	.0320	.0326	.0332	.0339	.0345	.0351	.0358
11	.0364	.0371	.0377	.0384	.0391	.0398	.0404	.0411	.0418	.0425
12	.0432	.0439	.0447	.0454	.0461	.0469	.0476	.0483	.0491	.0499
13	.0506	.0514	.0522	.0529	.0537	.0545	.0553	.0561	.0569	.0577
14	.0585	.0594	.0602	.0610	.0619	.0627	.0635	.0644	.0653	.0662
15	.0670	.0679	.0687	.0696	.0705	.0714	.0723	.0732	.0741	.0751
16	.0760	.0769	.0778	.0788	.0797	.0807	.0816	.0826	.0835	.0845
17	.0855	.0864	.0875	.0884	.0894	.0904	.0914	.0924	.0935	.0945
18	.0955	.0965	.0975	.0986	.0997	.1007	.1017	.1028	.1039	.1049
19	.1060	.1071	.1081	.1092	.1103	.1115	.1125	.1137	.1148	.1159
20	.1170	.1181	.1192	.1203	.1215	.1226	.1238	.1250	.1261	.1272
21	.1284	.1296	.1308	.1320	.1331	.1344	.1355	.1367	.1379	.1391
22	.1404	.1415	.1428	.1440	.1452	.1464	.1477	.1490	.1502	.1514
23	.1527	.1540	.1552	.1565	.1578	.1590	.1603	.1616	.1629	.1641
24	.1654	.1667	.1680	.1694	.1707	.1720	.1733	.1746	.1760	.1773
25	.1786	.1800	.1813	.1826	.1840	.1854	.1867	.1880	.1894	.1908
26	.1921	.1936	.1949	.1963	.1977	.1991	.2005	.2020	.2033	.2047
27	.2061	.2075	.2089	.2104	.2117	.2132	.2147	.2161	.2175	.2190
28	.2204	.2218	.2233	.2248	.2263	.2277	.2292	.2306	.2321	.2336
29	.2351	.2365	.2380	.2394	.2410	.2424	.2440	.2455	.2469	.2485
30	.2500	.2515	.2530	.2546	.2561	.2576	.2592	.2606	.2622	.2638
31	.2652	.2668	.2684	.2699	.2714	.2730	.2745	.2760	.2777	.2793
32	.2809	.2824	.2839	.2855	.2871	.2887	.2903	.2919	.2935	.2950
33	.2966	.2983	.2998	.3015	.3030	.3046	.3062	.3079	.3095	.3110
34	.3128	.3143	.3159	.3175	.3192	.3208	.3224	.3240	.3257	.3272
35	.3290	.3307	.3322	.3339	.3356	.3373	.3389	.3406	.3421	.3439
36	.3455	.3472	.3488	.3504	.3522	.3538	.3552	.3571	.3588	.3606
37	.3622	.3639	.3656	.3673	.3690	.3705	.3722	.3739	.3757	.3774
38	.3790	.3807	.3825	.3841	.3858	.3874	.3892	.3908	.3926	.3943
39	.3961	.3978	.3994	.4012	.4029	.4046	.4063	.4079	.4098	.4115
40	.4133	.4150	.4167	.4184	.4202	.4217	.4235	.4252	.4270	.4288
41	.4303	.4321	.4339	.4355	.4373	.4391	.4408	.4426	.4442	.4461
42	.4477	.4496	.4512	.4529	.4548	.4565	.4582	.4598	.4617	.4634
43	.4651	.4669	.4686	.4703	.4721	.4738	.4756	.4773	.4791	.4808
44	.4826	.4844	.4860	.4878	.4896	.4914	.4930	.4948	.4966	.4982

SIN² θ FOR SPECTROPHOTOMETER (CONTINUED)

°	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
45	.5000	.5017	.5035	.5051	.5070	.5086	.5105	.5122	.5141	.5157
46	.5174	.5193	.5210	.5226	.5243	.5263	.5280	.5297	.5314	.5331
47	.5351	.5365	.5383	.5400	.5418	.5435	.5453	.5470	.5488	.5506
48	.5523	.5541	.5557	.5575	.5593	.5611	.5626	.5644	.5662	.5678
49	.5696	.5712	.5731	.5746	.5765	.5781	.5800	.5816	.5834	.5851
50	.5869	.5886	.5902	.5921	.5937	.5954	.5970	.5990	.6006	.6023
51	.6046	.6056	.6073	.6090	.6107	.6124	.6140	.6157	.6174	.6192
52	.6209	.6226	.6243	.6260	.6280	.6295	.6310	.6327	.6345	.6362
53	.6377	.6394	.6412	.6430	.6445	.6462	.6477	.6495	.6513	.6528
54	.6546	.6561	.6580	.6595	.6610	.6628	.6644	.6662	.6677	.6693
55	.6710	.6727	.6742	.6759	.6775	.6792	.6808	.6823	.6841	.6856
56	.6872	.6890	.6906	.6922	.6937	.6953	.6970	.6986	.7002	.7018
57	.7034	.7050	.7065	.7081	.7097	.7114	.7129	.7145	.7160	.7176
58	.7191	.7208	.7223	.7238	.7254	.7270	.7284	.7301	.7316	.7332
59	.7347	.7362	.7377	.7393	.7408	.7423	.7439	.7454	.7470	.7485
60	.7501	.7514	.7530	.7544	.7560	.7575	.7589	.7605	.7621	.7635
61	.7649	.7665	.7679	.7693	.7707	.7723	.7737	.7752	.7768	.7782
62	.7796	.7811	.7825	.7840	.7854	.7869	.7881	.7896	.7910	.7925
63	.7940	.7952	.7967	.7982	.7995	.8009	.8022	.8037	.8050	.8065
64	.8078	.8093	.8106	.8119	.8134	.8147	.8160	.8173	.8187	.8200
65	.8215	.8228	.8241	.8255	.8268	.8279	.8293	.8306	.8320	.8333
66	.8346	.8360	.8372	.8385	.8397	.8410	.8424	.8435	.8449	.8461
67	.8474	.8486	.8498	.8511	.8523	.8535	.8549	.8561	.8572	.8584
68	.8596	.8608	.8622	.8634	.8646	.8658	.8670	.8680	.8692	.8704
69	.8716	.8728	.8740	.8750	.8762	.8774	.8784	.8796	.8808	.8819
70	.8831	.8841	.8853	.8863	.8876	.8886	.8896	.8908	.8919	.8929
71	.8939	.8952	.8962	.8972	.8983	.8993	.9003	.9014	.9024	.9034
72	.9045	.9055	.9066	.9076	.9087	.9095	.9105	.9116	.9126	.9135
73	.9145	.9156	.9164	.9175	.9183	.9194	.9203	.9213	.9221	.9230
74	.9241	.9249	.9258	.9268	.9277	.9285	.9294	.9305	.9313	.9322
75	.9330	.9339	.9348	.9356	.9365	.9374	.9382	.9391	.9397	.9406
76	.9415	.9423	.9432	.9438	.9447	.9456	.9463	.9471	.9478	.9486
77	.9493	.9502	.9508	.9517	.9524	.9532	.9539	.9546	.9554	.9561
78	.9568	.9574	.9581	.9590	.9596	.9603	.9609	.9616	.9623	.9629
79	.9636	.9643	.9650	.9656	.9660	.9667	.9674	.9681	.9687	.9692
80	.9698	.9704	.9710	.9716	.9722	.9728	.9733	.9739	.9744	.9750
81	.9755	.9761	.9766	.9771	.9777	.9782	.9787	.9792	.9797	.9801
82	.9806	.9811	.9816	.9820	.9825	.9830	.9834	.9838	.9843	.9847
83	.9852	.9856	.9860	.9864	.9868	.9872	.9876	.9880	.9883	.9887
84	.9891	.9894	.9898	.9901	.9905	.9908	.9912	.9915	.9918	.9921
85	.9924	.9927	.9930	.9933	.9936	.9938	.9941	.9944	.9946	.9949
86	.9951	.9954	.9956	.9958	.9960	.9963	.9965	.9967	.9960	.9971
87	.9973	.9974	.9976	.9978	.9979	.9981	.9983	.9984	.9985	.9987
88	.9988	.9989	.9990	.9991	.9992	.9993	.9994	.9995	.9996	.9996
89	.9997	.9998	.9998	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000

NATURAL SINES 0°–45°									PROPORTIONAL PARTS (Add)								
°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
0	.0000	.0029	.0058	.0087	.0116	.0145	.0175	89	3	6	9	12	14	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	14	17	20	23	26
2	.0349	.0378	.0407	.0436	.0465	.0494	.0523	87	3	6	9	12	14	17	20	23	26
3	.0523	.0552	.0581	.0610	.0640	.0669	.0698	86	3	6	9	12	14	17	20	23	26
4	.0698	.0727	.0756	.0785	.0814	.0843	.0872	85	3	6	9	12	14	17	20	23	26
5	.0872	.0901	.0929	.0958	.0987	.1016	.1045	84	3	6	9	12	14	17	20	23	26
6	.1045	.1074	.1103	.1132	.1161	.1190	.1219	83	3	6	9	12	14	17	20	23	26
7	.1219	.1248	.1276	.1305	.1334	.1363	.1392	82	3	6	9	12	14	17	20	23	26
8	.1392	.1421	.1449	.1478	.1507	.1536	.1564	81	3	6	9	12	14	17	20	23	26
9	.1564	.1593	.1622	.1650	.1679	.1708	.1736	80	3	6	9	12	14	17	20	23	26
10	.1736	.1765	.1794	.1822	.1851	.1880	.1908	79	3	6	9	12	14	17	20	23	26
11	.1908	.1937	.1965	.1994	.2022	.2051	.2079	78	3	6	8	11	14	17	20	22	25
12	.2079	.2108	.2136	.2164	.2193	.2221	.2250	77	3	6	8	11	14	17	20	22	25
13	.2250	.2278	.2306	.2334	.2363	.2391	.2419	76	3	6	8	11	14	17	20	22	25
14	.2419	.2447	.2476	.2504	.2532	.2560	.2588	75	3	6	8	11	14	17	20	22	25
15	.2588	.2616	.2644	.2672	.2700	.2728	.2756	74	3	6	8	11	14	17	20	22	25
16	.2756	.2784	.2812	.2840	.2868	.2896	.2924	73	3	6	8	11	14	17	20	22	25
17	.2924	.2952	.2979	.3007	.3035	.3062	.3090	72	3	6	8	11	14	17	20	22	25
18	.3090	.3118	.3145	.3173	.3201	.3228	.3256	71	3	6	8	11	14	17	20	22	25
19	.3256	.3283	.3311	.3338	.3365	.3393	.3420	70	3	5	8	11	14	16	19	22	24
20	.3420	.3448	.3475	.3502	.3529	.3557	.3584	69	3	5	8	11	14	16	19	22	24
21	.3584	.3611	.3638	.3665	.3692	.3719	.3746	68	3	5	8	11	14	16	19	22	24
22	.3746	.3773	.3800	.3827	.3854	.3881	.3907	67	3	5	8	11	14	16	19	22	24
23	.3907	.3934	.3961	.3987	.4014	.4041	.4067	66	3	5	8	11	14	16	19	22	24
24	.4067	.4094	.4120	.4147	.4173	.4200	.4226	65	3	5	8	11	14	16	19	22	24
25	.4226	.4253	.4279	.4305	.4331	.4358	.4384	64	3	5	8	10	13	16	18	21	23
26	.4384	.4410	.4436	.4462	.4488	.4514	.4540	63	3	5	8	10	13	16	18	21	23
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62	3	5	8	10	13	16	18	21	23
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61	3	5	8	10	13	16	18	21	23
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60	3	5	8	10	13	15	18	20	23
30	.5000	.5025	.5050	.5075	.5100	.5125	.5150	59	3	5	8	10	13	15	18	20	23
31	.5150	.5175	.5200	.5225	.5250	.5275	.5299	58	3	5	8	10	13	15	18	20	23
32	.5299	.5324	.5348	.5373	.5398	.5422	.5446	57	3	5	8	10	13	15	18	20	23
33	.5446	.5471	.5495	.5519	.5544	.5568	.5592	56	3	5	7	10	12	14	17	19	22
34	.5592	.5616	.5640	.5664	.5688	.5712	.5736	55	3	5	7	10	12	14	17	19	22
35	.5736	.5760	.5783	.5807	.5831	.5854	.5878	54	2	5	7	10	12	14	17	19	22
36	.5878	.5901	.5925	.5948	.5972	.5995	.6018	53	2	5	7	9	12	14	16	18	21
37	.6018	.6041	.6065	.6088	.6111	.6134	.6157	52	2	5	7	9	12	14	16	18	21
38	.6157	.6180	.6202	.6225	.6248	.6271	.6293	51	2	5	7	9	12	14	16	18	21
39	.6293	.6316	.6338	.6361	.6383	.6406	.6428	50	2	4	7	9	11	13	15	18	20
40	.6428	.6450	.6472	.6494	.6517	.6539	.6561	49	2	4	7	9	11	13	15	18	20
41	.6561	.6583	.6604	.6626	.6648	.6670	.6691	48	2	4	7	9	11	13	15	18	20
42	.6691	.6713	.6734	.6756	.6777	.6799	.6820	47	2	4	6	8	11	13	15	17	19
43	.6820	.6841	.6862	.6884	.6905	.6926	.6947	46	2	4	6	8	11	13	15	17	19
44	.6947	.6967	.6988	.7009	.7030	.7050	.7071	45	2	4	6	8	11	13	15	17	19
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9
NATURAL COSINES 45°–90°									PROPORTIONAL PARTS (Subtract)								

LOGARITHMIC SINES 0°–45°

PROPORTIONAL PARTS (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
0	∞	$\bar{3}$.4637	.7648	.9408	*.0658*	*.1627*	*.2419*	89	Differences				3011–792				
1	$\bar{2}$.2419	.3088	.3668	.4179	.4637	.5050	.5428	88					669–378				
2	.5428	.5776	.6097	.6397	.6677	.6940	.7188	87					348–248				
3	.7188	.7423	.7645	.7857	.8059	.8251	.8436	86					235–185				
4	.8436	.8613	.8783	.8946	.9104	.9256	.9403	85					177–147				
5	$\bar{2}$.9403	.9545	.9682	.9816	.9945	*.0070*	*.0192*	84					142–122				
6	$\bar{1}$.0192	.0311	.0426	.0539	.0648	.0755	.0859	83					119–104				
7	.0859	.0961	.1060	.1157	.1252	.1345	.1436	82					102–91				
8	.1436	.1525	.1612	.1697	.1781	.1863	.1943	81					89–80				
9	.1943	.2022	.2100	.2176	.2251	.2324	.2397	80					79–73				
10	$\bar{1}$.2397	.2468	.2538	.2606	.2674	.2740	.2806	79	7	14	20	27	34	41	48	54	61
11	.2806	.2870	.2934	.2997	.3058	.3119	.3179	78	6	12	19	25	31	37	43	50	56
12	.3179	.3238	.3296	.3353	.3410	.3466	.3521	77	6	11	17	23	29	34	40	46	51
13	.3521	.3575	.3629	.3682	.3734	.3786	.3837	76	5	11	16	21	27	32	37	42	48
14	.3837	.3887	.3937	.3986	.4035	.4083	.4130	75	5	10	15	20	25	29	34	39	44
15	$\bar{1}$.4130	.4177	.4223	.4269	.4314	.4359	.4403	74	5	9	14	18	23	27	32	37	41
16	.4403	.4447	.4491	.4533	.4576	.4618	.4659	73	4	9	13	17	21	26	30	34	39
17	.4659	.4700	.4741	.4781	.4821	.4861	.4900	72	4	8	12	16	20	24	28	32	36
18	.4900	.4939	.4977	.5015	.5052	.5090	.5126	71	4	8	11	15	19	23	27	30	34
19	.5126	.5163	.5199	.5235	.5270	.5306	.5341	70	4	7	10	14	18	22	25	29	32
20	$\bar{1}$.5341	.5375	.5409	.5443	.5477	.5510	.5543	69	3	7	10	14	17	20	24	27	31
21	.5543	.5576	.5609	.5641	.5673	.5704	.5736	68	3	6	10	13	16	19	22	26	29
22	.5736	.5767	.5798	.5828	.5859	.5889	.5919	67	3	6	9	12	15	18	21	24	28
23	.5919	.5948	.5978	.6007	.6036	.6065	.6093	66	3	6	9	12	15	17	20	23	26
24	.6093	.6121	.6149	.6177	.6205	.6232	.6259	65	3	6	8	11	14	16	19	22	25
25	$\bar{1}$.6259	.6286	.6313	.6340	.6366	.6392	.6418	64	3	5	8	11	13	16	19	21	24
26	.6418	.6444	.6470	.6495	.6521	.6546	.6570	63	3	5	8	10	13	15	18	20	23
27	.6570	.6595	.6620	.6644	.6668	.6692	.6716	62	2	5	7	10	12	15	17	19	22
28	.6716	.6740	.6763	.6787	.6810	.6833	.6856	61	2	5	7	9	12	14	16	19	21
29	.6856	.6878	.6901	.6923	.6946	.6968	.6990	60	2	4	7	9	11	13	16	18	20
30	$\bar{1}$.6990	.7012	.7033	.7055	.7076	.7097	.7118	59	2	4	6	9	11	13	15	17	19
31	.7118	.7139	.7160	.7181	.7201	.7222	.7242	58	2	4	6	8	10	12	14	16	19
32	.7242	.7262	.7282	.7302	.7322	.7342	.7361	57	2	4	6	8	10	12	14	16	18
33	.7361	.7380	.7400	.7419	.7438	.7457	.7476	56	2	4	6	8	10	11	13	15	17
34	.7476	.7494	.7513	.7531	.7550	.7568	.7586	55	2	4	6	7	9	11	13	15	16
35	$\bar{1}$.7586	.7604	.7622	.7640	.7657	.7675	.7692	54	2	4	5	7	9	11	12	14	16
36	.7692	.7710	.7727	.7744	.7761	.7778	.7795	53	2	3	5	7	9	10	12	14	15
37	.7795	.7811	.7828	.7844	.7861	.7877	.7893	52	2	3	5	7	8	10	11	13	15
38	.7893	.7910	.7926	.7941	.7957	.7973	.7989	51	2	3	5	6	8	10	11	13	14
39	.7989	.8004	.8020	.8035	.8050	.8066	.8081	50	2	3	5	6	8	9	11	12	14
40	$\bar{1}$.8081	.8096	.8111	.8125	.8140	.8155	.8169	49	1	3	4	6	7	9	10	12	13
41	.8169	.8184	.8198	.8213	.8227	.8241	.8255	48	1	3	4	6	7	9	10	11	13
42	.8255	.8269	.8283	.8297	.8311	.8324	.8338	47	1	3	4	6	7	8	10	11	13
43	.8338	.8351	.8365	.8378	.8391	.8405	.8418	46	1	3	4	5	7	8	9	11	12
44	.8418	.8431	.8444	.8457	.8469	.8482	.8495	45	1	3	4	5	6	8	9	10	12
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

LOGARITHMIC COSINES 45°–90°

PROPORTIONAL PARTS (Subtract)

NATURAL SINES 45°–90° | PROPORTIONAL PARTS (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
45	.7071	.7092	.7112	.7133	.7153	.7173	.7193	44	2	4	6	8	10	12	14	16	18
46	.7193	.7214	.7234	.7254	.7274	.7294	.7314	43	2	4	6	8	10	12	14	16	18
47	.7314	.7333	.7353	.7373	.7392	.7412	.7431	42	2	4	6	8	10	12	14	15	18
48	.7431	.7451	.7470	.7490	.7509	.7528	.7547	41	2	4	6	8	10	11	13	15	17
49	.7547	.7566	.7585	.7604	.7623	.7642	.7660	40	2	4	6	8	9	11	13	15	17
50	.7660	.7679	.7698	.7716	.7735	.7753	.7771	39	2	4	6	7	9	11	13	15	17
51	.7771	.7790	.7808	.7826	.7844	.7862	.7880	38	2	4	5	7	9	11	13	15	16
52	.7880	.7898	.7916	.7934	.7951	.7969	.7986	37	2	3	5	7	9	10	12	14	16
53	.7986	.8004	.8021	.8039	.8056	.8073	.8090	36	2	3	5	7	9	10	12	14	16
54	.8090	.8107	.8124	.8141	.8158	.8175	.8192	35	2	3	5	7	8	10	12	14	15
55	.8192	.8208	.8225	.8241	.8258	.8274	.8290	34	2	3	5	7	8	10	11	13	15
56	.8290	.8307	.8323	.8339	.8355	.8371	.8387	33	2	3	5	6	8	10	11	13	15
57	.8387	.8403	.8418	.8434	.8450	.8465	.8480	32	2	3	5	6	8	9	11	12	14
58	.8480	.8496	.8511	.8526	.8542	.8557	.8572	31	2	3	5	6	8	9	11	12	14
59	.8572	.8587	.8601	.8616	.8631	.8646	.8660	30	1	3	4	6	7	9	10	12	13
60	.8660	.8675	.8689	.8704	.8718	.8732	.8746	29	1	3	4	6	7	9	10	11	13
61	.8746	.8760	.8774	.8788	.8802	.8816	.8829	28	1	3	4	6	7	8	10	11	12
62	.8829	.8843	.8857	.8870	.8884	.8897	.8910	27	1	3	4	5	7	8	9	11	12
63	.8910	.8923	.8936	.8949	.8962	.8975	.8988	26	1	3	4	5	6	8	9	10	12
64	.8988	.9001	.9013	.9026	.9038	.9051	.9063	25	1	3	4	5	6	7	9	10	11
65	.9063	.9075	.9088	.9100	.9112	.9124	.9135	24	1	2	4	5	6	7	9	10	11
66	.9135	.9147	.9159	.9171	.9182	.9194	.9205	23	1	2	4	5	6	7	8	9	11
67	.9205	.9216	.9228	.9239	.9250	.9261	.9272	22	1	2	3	4	6	7	8	9	10
68	.9272	.9283	.9293	.9304	.9315	.9325	.9336	21	1	2	3	4	5	6	7	9	10
69	.9336	.9346	.9356	.9367	.9377	.9387	.9397	20	1	2	3	4	5	6	7	8	9
70	.9397	.9407	.9417	.9426	.9436	.9446	.9455	19	1	2	3	4	5	6	7	8	9
71	.9455	.9465	.9474	.9483	.9492	.9502	.9511	18	1	2	3	4	5	6	7	7	8
72	.9511	.9520	.9528	.9537	.9546	.9555	.9563	17	1	2	3	3	4	5	6	7	8
73	.9563	.9572	.9580	.9588	.9596	.9605	.9613	16	1	2	2	3	4	5	6	7	7
74	.9613	.9621	.9628	.9636	.9644	.9652	.9659	15	1	2	2	3	4	5	5	6	7
75	.9659	.9667	.9674	.9681	.9689	.9696	.9703	14	1	1	2	3	4	4	5	6	7
76	.9703	.9710	.9717	.9724	.9730	.9737	.9744	13	1	1	2	3	4	4	5	6	6
77	.9744	.9750	.9757	.9763	.9769	.9775	.9781	12	1	1	2	3	3	4	4	5	6
78	.9781	.9787	.9793	.9799	.9805	.9811	.9816	11	1	1	2	2	3	4	4	5	5
79	.9816	.9822	.9827	.9833	.9838	.9843	.9848	10	1	1	2	2	3	3	4	4	5
80	.9848	.9853	.9858	.9863	.9868	.9872	.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	4	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	1	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	.9962	.9964	.9967	.9969	.9971	.9974	.9976	4	0	1	1	1	1	2	2	2	2
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3	0	0	1	1	1	1	2	2	2
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2	0	0	1	1	1	1	1	1	2
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1	0	0	0	0	0	1	1	1	1
89	.9998	.9999	.9999	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	1	1	1	1
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

NATURAL COSINES 0°–45° | PROPORTIONAL PARTS (Subtract)

LOGARITHMIC SINES 45°–90°

PROPORTIONAL PARTS (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
45	$\bar{1}$.8495	.8507	.8520	.8532	.8545	.8557	.8569	44	1	2	4	5	6	7	9	10	11
46	.8569	.8582	.8594	.8606	.8618	.8629	.8641	43	1	2	4	5	6	7	8	10	11
47	.8641	.8653	.8665	.8676	.8688	.8699	.8711	42	1	2	4	5	6	7	8	9	11
48	.8711	.8722	.8733	.8745	.8756	.8767	.8778	41	1	2	3	4	6	7	8	9	10
49	.8778	.8789	.8800	.8810	.8821	.8832	.8843	40	1	2	3	4	5	7	8	9	10
50	$\bar{1}$.8843	.8853	.8864	.8874	.8884	.8895	.8905	39	1	2	3	4	5	6	7	9	10
51	.8905	.8915	.8925	.8935	.8945	.8955	.8965	38	1	2	3	4	5	6	7	8	9
52	.8965	.8975	.8985	.8995	.9004	.9014	.9023	37	1	2	3	4	5	6	7	8	9
53	.9023	.9033	.9042	.9052	.9061	.9070	.9080	36	1	2	3	4	5	6	7	8	9
54	.9080	.9089	.9098	.9107	.9116	.9125	.9134	35	1	2	3	4	4	5	6	7	8
55	$\bar{1}$.9134	.9142	.9151	.9160	.9169	.9177	.9186	34	1	2	3	3	4	5	6	7	8
56	.9186	.9194	.9203	.9211	.9219	.9228	.9236	33	1	2	2	3	4	5	6	7	7
57	.9236	.9244	.9252	.9260	.9268	.9276	.9284	32	1	2	2	3	4	5	6	6	7
58	.9284	.9292	.9300	.9308	.9315	.9323	.9331	31	1	2	2	3	4	5	6	6	7
59	.9331	.9338	.9346	.9353	.9361	.9368	.9375	30	1	2	2	3	4	5	5	6	7
60	$\bar{1}$.9375	.9383	.9390	.9397	.9404	.9411	.9418	29	1	1	2	3	4	4	5	6	6
61	.9418	.9425	.9432	.9439	.9446	.9453	.9459	28	1	1	2	3	3	4	5	5	6
62	.9459	.9466	.9473	.9479	.9486	.9492	.9499	27	1	1	2	3	3	4	5	5	6
63	.9499	.9505	.9512	.9518	.9524	.9530	.9537	26	1	1	2	3	3	4	4	5	6
64	.9537	.9543	.9549	.9555	.9561	.9567	.9573	25	1	1	2	2	3	4	4	5	5
65	$\bar{1}$.9573	.9579	.9584	.9590	.9596	.9602	.9607	24	1	1	2	2	3	3	4	5	5
66	.9607	.9613	.9618	.9624	.9629	.9635	.9640	23	1	1	2	2	3	3	4	4	5
67	.9640	.9646	.9651	.9656	.9661	.9667	.9672	22	1	1	2	2	3	3	4	4	5
68	.9672	.9677	.9682	.9687	.9692	.9697	.9702	21	1	1	2	2	3	3	4	4	5
69	.9702	.9706	.9711	.9716	.9721	.9725	.9730	20	0	1	1	2	2	3	3	4	4
70	$\bar{1}$.9730	.9734	.9739	.9743	.9748	.9752	.9757	19	0	1	1	2	2	3	3	4	4
71	.9757	.9761	.9765	.9770	.9774	.9778	.9782	18	0	1	1	2	2	2	3	3	4
72	.9782	.9786	.9790	.9794	.9798	.9802	.9806	17	0	1	1	2	2	2	3	3	4
73	.9806	.9810	.9814	.9817	.9821	.9825	.9828	16	0	1	1	1	2	2	3	3	3
74	.9828	.9832	.9836	.9839	.9843	.9846	.9849	15	0	1	1	1	2	2	2	3	3
75	$\bar{1}$.9849	.9853	.9856	.9859	.9863	.9866	.9869	14	0	1	1	1	2	2	2	3	3
76	.9869	.9872	.9875	.9878	.9881	.9884	.9887	13	0	1	1	1	2	2	2	2	3
77	.9887	.9890	.9893	.9896	.9899	.9901	.9904	12	0	1	1	1	1	2	2	2	3
78	.9904	.9907	9909	.9912	.9914	.9917	.9919	11	0	1	1	1	1	2	2	2	3
79	.9919	.9922	.9924	.9927	.9929	.9931	.9934	10	0	1	1	1	1	2	2	2	3
80	$\bar{1}$.9934	.9936	.9938	.9940	.9942	.9944	.9946	9					2				
81	.9946	.9948	.9950	.9952	.9954	.9956	.9958	8					2				
82	.9958	.9959	.9961	.9963	.9964	.9966	.9968	7					1–2				
83	.9968	.9969	.9971	.9972	.9973	.9975	.9976	6					1–2				
84	.9976	.9977	.9979	.9980	.9981	.9982	.9983	5					1–2				
85	$\bar{1}$.9983	.9985	.9986	.9987	.9988	.9989	.9989	4	Differences				0–2				
86	.9989	.9990	.9991	.9992	.9993	.9993	.9994	3					0–1				
87	.9994	.9995	.9995	.9996	.9996	.9997	.9997	2					0–1				
88	.9997	.9998	.9998	.9999	.9999	.9999	.9999	1					0–1				
89	.9999	.0000	.0000	.0000	.0000	.0000	.0000	0					0–1				
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

LOGARITHMIC COSINES 0°–45°

PROPORTIONAL PARTS (Subtract)

NATURAL TANGENTS 0°–45°

PROPORTIONAL PARTS (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
0	.0000	.0029	.0058	.0087	.0116	.0145	.0175	89	3	6	9	12	15	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	15	17	20	23	26
2	.0349	.0378	.0407	.0437	.0466	.0495	.0524	87	3	6	9	12	15	17	20	23	26
3	.0524	.0553	.0582	.0612	.0641	.0670	.0699	86	3	6	9	12	15	17	20	23	26
4	.0699	.0729	.0758	.0787	.0816	.0846	.0875	85	3	6	9	12	15	18	21	24	26
5	.0875	.0904	.0934	.0963	.0992	.1022	.1051	84	3	6	9	12	15	18	21	24	26
6	.1051	.1080	.1110	.1139	.1169	.1198	.1228	83	3	6	9	12	15	18	21	24	27
7	.1228	.1257	.1287	.1317	.1346	.1376	.1405	82	3	6	9	12	15	18	21	24	27
8	.1405	.1435	.1465	.1495	.1524	.1554	.1584	81	3	6	9	12	15	18	21	24	27
9	.1584	.1614	.1644	.1673	.1703	.1733	.1763	80	3	6	9	12	15	18	21	24	27
10	.1763	.1793	.1823	.1853	.1883	.1914	.1944	79	3	6	9	12	15	18	21	24	27
11	.1944	.1974	.2004	.2035	.2065	.2095	.2126	78	3	6	9	12	15	18	21	24	27
12	.2126	.2156	.2186	.2217	.2247	.2278	.2309	77	3	6	9	12	15	18	22	25	28
13	.2309	.2339	.2370	.2401	.2432	.2462	.2493	76	3	6	9	12	15	18	22	25	28
14	.2493	.2524	.2555	.2586	.2617	.2648	.2679	75	3	6	9	12	16	19	22	25	28
15	.2679	.2711	.2742	.2773	.2805	.2836	.2867	74	3	6	9	12	16	19	22	25	28
16	.2867	.2899	.2931	.2962	.2994	.3026	.3057	73	3	6	10	13	16	19	22	26	29
17	.3057	.3089	.3121	.3153	.3185	.3217	.3249	72	3	6	10	13	16	19	22	26	29
18	.3249	.3281	.3314	.3346	.3378	.3411	.3443	71	3	6	10	13	16	19	23	26	29
19	.3443	.3476	.3508	.3541	.3574	.3607	.3640	70	3	7	10	13	16	20	23	26	30
20	.3640	.3673	.3706	.3739	.3772	.3805	.3839	69	3	7	10	13	17	20	23	26	30
21	.3839	.3872	.3906	.3939	.3973	.4006	.4040	68	3	7	10	13	17	20	23	27	30
22	.4040	.4074	.4108	.4142	.4176	.4210	.4245	67	3	7	10	14	17	21	24	27	31
23	.4245	.4279	.4314	.4348	.4383	.4417	.4452	66	3	7	10	14	17	21	24	28	31
24	.4452	.4487	.4522	.4557	.4592	.4628	.4663	65	4	7	11	14	18	21	25	28	32
25	.4663	.4699	.4734	.4770	.4806	.4841	.4877	64	4	7	11	14	18	21	25	29	32
26	.4877	.4913	.4950	.4986	.5022	.5059	.5095	63	4	7	11	15	18	22	25	29	33
27	.5095	.5132	.5169	.5206	.5243	.5280	.5317	62	4	7	11	15	19	22	26	30	33
28	.5317	.5354	.5392	.5430	.5467	.5505	.5543	61	4	8	11	15	19	23	26	30	34
29	.5543	.5581	.5619	.5658	.5696	.5735	.5774	60	4	8	12	15	19	23	27	31	35
30	.5774	.5812	.5851	.5890	.5930	.5969	.6009	59	4	8	12	16	20	23	27	31	35
31	.6009	.6048	.6088	.6128	.6168	.6208	.6249	58	4	8	12	16	20	24	28	32	36
32	.6249	.6289	.6330	.6371	.6412	.6453	.6494	57	4	8	12	16	20	24	28	33	37
33	.6494	.6536	.6577	.6619	.6661	.6703	.6745	56	4	8	13	17	21	25	29	34	38
34	.6745	.6787	.6830	.6873	.6916	.6959	.7002	55	4	9	13	17	21	26	30	34	39
35	.7002	.7046	.7089	.7133	.7177	.7221	.7265	54	4	9	13	18	22	26	31	35	40
36	.7265	.7310	.7355	.7400	.7445	.7490	.7536	53	5	9	14	18	23	27	32	36	41
37	.7536	.7581	.7627	.7673	.7720	.7766	.7813	52	5	9	14	18	23	28	32	37	42
38	.7813	.7860	.7907	.7954	.8002	.8050	.8098	51	5	10	14	19	24	29	33	38	43
39	.8098	.8146	.8195	.8243	.8292	.8342	.8391	50	5	10	15	20	24	29	34	39	44
40	.8391	.8441	.8491	.8541	.8591	.8642	.8693	49	5	10	15	20	25	30	35	40	45
41	.8693	.8744	.8796	.8847	.8899	.8952	.9004	48	5	10	16	21	26	31	36	42	47
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47	5	11	16	21	27	32	37	43	48
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46	6	11	17	22	28	33	39	44	50
44	.9657	.9713	.9770	.9827	.9884	.9942	1.0000	45	6	11	17	23	29	34	40	46	51
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

NATURAL COTANGENTS 45°–90°

PROPORTIONAL PARTS (Subtract)

Logarithmic Tangents 0°–45°

Proportional Parts (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
0	− ∞	$\bar{3}$.4637	.7648	.9409	*.0658*	*.1627*	*.2419*	89	Differences			3011–792					
1	$\bar{2}$.2419	.3089	.3669	.4181	.4638	.5053	.5431	88				670–378					
2	.5431	.5779	.6101	.6401	.6682	.6945	.7194	87				348–249					
3	.7194	.7429	.7652	.7865	.8067	.8261	.8446	86				235–185					
4	.8446	.8624	.8795	.8960	.9118	.9272	.9420	85				170–148					
5	$\bar{2}$.9420	.9563	.9701	.9836	.9966	*.0093*	*.0216*	84				143–124					
6	$\bar{1}$.0216	.0336	.0453	.0567	.0678	.0786	.0891	83				120–105					
7	.0891	.0995	.1096	.1194	.1291	.1385	.1478	82				104–93					
8	.1478	.1569	.1658	.1745	.1831	.1915	.1997	81				91–82					
9	.1997	.2078	.2158	.2236	.2313	.2389	.2463	80				81–74					
10	$\bar{1}$.2463	.2536	.2609	.2680	.2750	.2819	.2887	79	7	14	21	28	36	43	50	57	64
11	.2887	.2954	.3020	.3085	.3149	.3212	.3275	78	7	13	20	26	33	39	46	52	59
12	.3275	.3336	.3397	.3458	.3517	.3576	.3634	77	6	12	18	24	30	36	42	48	54
13	.3634	.3691	.3748	.3804	.3859	.3914	.3968	76	6	11	17	22	28	33	39	44	50
14	.3968	.4021	.4074	.4127	.4178	.4230	.4281	75	5	10	16	21	26	31	36	42	47
15	$\bar{1}$.4281	.4331	.4381	.4430	.4479	.4527	.4575	74	5	10	15	20	25	29	34	39	44
16	.4575	.4622	.4669	.4716	.4762	.4808	.4853	73	5	9	14	19	24	28	33	38	42
17	.4853	.4898	.4943	.4987	.5031	.5075	.5118	72	4	9	13	18	22	27	31	35	40
18	.5118	.5161	.5203	.5245	.5287	.5329	.5370	71	4	8	13	17	21	25	29	34	38
19	.5370	.5411	.5451	.5491	.5531	.5571	.5611	70	4	8	12	16	20	24	28	32	36
20	$\bar{1}$.5611	.5650	.5689	.5727	.5766	.5804	.5842	69	4	8	12	15	19	23	27	31	35
21	.5842	.5879	.5917	.5954	.5991	.6028	.6064	68	4	7	11	15	18	22	26	30	33
22	.6064	.6100	.6136	.6172	.6208	.6243	.6279	67	4	7	11	14	18	22	25	29	32
23	.6279	.6314	.6348	.6383	.6417	.6452	.6486	66	3	7	10	14	17	21	24	28	31
24	.6486	.6520	.6553	.6587	.6620	.6654	.6687	65	3	7	10	13	17	20	23	27	30
25	$\bar{1}$.6687	.6720	.6752	.6785	.6817	.6850	.6882	64	3	7	10	13	16	20	23	26	29
26	.6882	.6914	.6946	.6977	.7009	.7040	.7072	63	3	6	10	13	16	19	22	26	29
27	.7072	.7103	.7134	.7165	.7196	.7226	.7257	62	3	6	9	12	15	18	21	25	28
28	.7257	.7287	.7317	.7348	.7378	.7408	.7438	61	3	6	9	12	15	18	21	24	27
29	.7438	.7467	.7497	.7526	.7556	.7585	.7614	60	3	6	9	12	15	18	21	23	26
30	$\bar{1}$.7614	.7644	.7673	.7701	.7730	.7759	.7788	59	3	6	9	12	14	17	20	23	26
31	.7788	.7816	.7845	.7873	.7902	.7930	.7958	58	3	6	9	11	14	17	20	23	25
32	.7958	.7986	.8014	.8042	.8070	.8097	.8125	57	3	6	8	11	14	17	20	22	25
33	.8125	.8153	.8180	.8208	.8235	.8263	.8290	56	3	6	8	11	14	17	19	22	25
34	.8290	.8317	.8344	.8371	.8398	.8425	.8452	55	3	5	8	11	13	16	19	22	24
35	$\bar{1}$.8452	.8479	.8506	.8533	.8559	.8586	.8613	54	3	5	8	11	13	16	19	21	24
36	.8613	.8639	.8666	.8692	.8718	.8745	.8771	53	3	5	8	11	13	16	18	21	24
37	.8771	.8797	.8824	.8850	.8876	.8902	.8928	52	3	5	8	10	13	16	18	21	24
38	.8928	.8954	.8980	.9006	.9032	.9058	.9084	51	3	5	8	10	13	16	18	21	23
39	.9084	.9110	.9135	.9161	.9187	.9212	.9238	50	3	5	8	10	13	15	18	21	23
40	$\bar{1}$.9238	.9264	.9289	.9315	.9341	.9366	.9392	49	3	5	8	10	13	15	18	21	23
41	.9392	.9417	.9443	.9468	.9494	.9519	.9544	48	3	5	8	10	13	15	18	20	23
42	.9544	.9570	.9595	.9621	.9646	.9671	.9697	47	3	5	8	10	13	15	18	20	23
43	.9697	.9722	.9747	.9772	.9798	.9823	.9848	46	3	5	8	10	13	15	18	20	23
44	.9848	.9874	.9899	.9924	.9949	.9975	.0000	45	3	5	8	10	13	15	18	20	23
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

Logarithmic Cotangents 45°–90°

Proportional Parts (Subtract)

NATURAL TANGENTS 45°–90°

°	0′	10′	20′	30′	40′	50′		°
		Diff.	Diff.	Diff.	Diff.	Diff.	Diff.	
45	1.0000	58 .0058	59 .0117	59 .0176	59 .0235	60 .0295	60 .0355	44
46	.0355	61 .0416	61 .0477	61 .0538	61 .0599	62 .0661	62 .0723	43
47	.0723	63 .0786	64 .0850	63 .0913	64 .0977	64 .1041	65 .1106	42
48	.1106	65 .1171	66 .1237	66 .1303	66 .1369	67 .1436	68 .1504	41
49	.1504	68 .1572	68 .1640	69 .1709	69 .1778	69 .1847	71 .1918	40
50	1.1918	70 .1988	71 .2059	72 .2131	72 .2203	73 .2276	73 .2349	39
51	.2349	74 .2423	74 .2497	75 .2572	75 .2647	76 .2723	76 .2799	38
52	.2799	77 .2876	78 .2954	78 .3032	79 .3111	79 .3190	80 .3270	37
53	.3270	81 .3351	81 .3432	82 .3514	83 .3597	83 .3680	84 .3764	36
54	.3764	84 .3848	85 .3933	86 .4019	87 .4106	87 .4193	88 .4281	35
55	1.4281	89 .4370	90 .4460	90 .4550	91 .4641	92 .4733	93 .4826	34
56	.4826	93 .4919	94 .5013	95 .5108	96 .5204	97 .5301	98 .5399	33
57	.5399	98 .5497	100 .5597	100 .5697	101 .5798	102 .5900	103 .6003	32
58	.6003	104 .6107	105 .6212	107 .6319	107 .6426	108 .6534	109 .6643	31
59	.6643	110 .6753	111 .6864	113 .6977	113 .7090	115 .7205	116 .7321	30
60	1.7321	116 .7437	119 .7556	119 .7675	121 .7796	121 .7917	123 .8040	29
61	.8040	125 .8165	126 .8291	127 .8418	128 .8546	130 .8676	131 .8807	28
62	.8807	133 .8940	134 .9074	136 .9210	137 .9347	139 .9486	140 .9626	27
63	.9626	142 .9768	144 .9912	145 *.0057*	147 *.0204*	149 *.0353*	150 *.0503*	26
64	2.0503	152 .0655	154 .0809	156 .0965	158 .1123	160 .1283	162 .1445	25
65	2.1445	164 .1609	166 .1775	168 .1943	170 .2113	173 .2286	174 .2460	24
66	.2460	177 .2637	180 .2817	181 .2998	185 .3183	186 .3369	190 .3559	23
67	.3559	191 .3750	195 .3945	197 .4142	200 .4342	203 .4545	206 .4751	22
68	.4751	209 .4960	212 .5172	214 .5386	219 .5605	221 .5826	225 .6051	21
69	.6051	228 .6279	232 .6511	235 .6746	239 .6985	243 .7228	247 .7475	20
70	2.7475	250 .7725	255 .7980	259 .8239	263 .8502	268 .8770	272 .9042	19
71	.9042	277 .9319	281 .9600	287 .9887	291 *.0178*	297 *.0475*	302 *.0777*	18
72	3.0777	307 .1084	313 .1397	319 .1716	325 .2041	330 .2371	338 .2709	17
73	.2709	343 .3052	350 .3402	357 .3759	365 .4124	371 .4495	379 .4874	16
74	.4874	387 .5261	395 .5656	403 .6059	411 .6470	421 .6891	430 .7321	15
75	3.7321	439 .7760	448 .8208	459 .8667	469 .9136	481 .9617	491 *.0108*	14
76	4.0108	503 .0611	515 .1126	527 .1653	540 .2193	554 .2747	568 .3315	13
77	.3315	582 .3897	597 .4494	613 .5107	629 .5736	646 .6382	664 .7046	12
78	.7046	683 .7729	701 .8430	722 .9152	742 .9894	764 *.0658*	788 *.1446*	11
79	5.1446	811 .2257	836 .3093	862 .3955	890 .4845	919 .5764	949 .6713	10
80	5.6713	5.7694	5.8708	5.9758	6.0844	6.1970	6.3138	9
81	6.3138	6.4348	6.5606	6.6912	6.8269	6.9682	7.1154	8
82	7.1154	7.2687	7.4287	7.5958	7.7704	7.9530	8.1443	7
83	8.1443	8.3450	8.5555	8.7769	9.0098	9.2553	9.5144	6
84	9.5144	9.7882	10.078	10.385	10.712	11.059	11.430	5
85	11.430	11.826	12.251	12.706	13.197	13.727	14.301	4
86	14.301	14.924	15.605	16.350	17.169	18.075	19.081	3
87	19.081	20.206	21.470	22.904	24.542	26.432	28.636	2
88	28.636	31.242	34.368	38.188	42.964	49.104	57.290	1
89	57.290	68.750	85.940	114.59	171.89	343.77	∞	0
°		50′	40′	30′	20′	10′	0′	°

NATURAL COTANGENTS 0°–45°

LOGARITHMIC TANGENTS 45°–90° | PROPORTIONAL PARTS (Add)

°	0′	10′	20′	30′	40′	50′		°	1	2	3	4	5	6	7	8	9
45	0.0000	.0025	.0051	.0076	.0101	.0126	.0152	44	3	5	8	10	13	15	18	20	23
46	.0152	.0177	.0202	.0228	.0253	.0278	.0303	43	3	5	8	10	13	15	18	20	23
47	.0303	.0329	.0354	.0379	.0405	.0430	.0456	42	3	5	8	10	13	15	18	20	23
48	.0456	.0481	.0506	.0532	.0557	.0583	.0608	41	3	5	8	10	13	15	18	20	23
49	.0608	.0634	.0659	.0685	.0711	.0736	.0762	40	3	5	8	10	13	15	18	20	23
50	0.0762	.0788	.0813	.0839	.0865	.0890	.0916	39	3	5	8	10	13	15	18	20	23
51	.0916	.0942	.0968	.0994	.1020	.1046	.1072	38	3	5	8	10	13	16	18	21	23
52	.1072	.1098	.1124	.1150	.1176	.1203	.1229	37	3	5	8	10	13	16	18	21	23
53	.1229	.1255	.1282	.1308	.1334	.1361	.1387	36	3	5	8	11	13	16	18	21	24
54	.1387	.1414	.1441	.1467	.1494	.1521	.1548	35	3	5	8	11	13	16	19	21	24
55	0.1548	.1575	.1602	.1629	.1656	.1683	.1710	34	3	5	8	11	14	16	19	22	24
56	.1710	.1737	.1765	.1792	.1820	.1847	.1875	33	3	6	8	11	14	17	19	22	25
57	.1875	.1903	.1930	.1958	.1986	.2014	.2042	32	3	6	8	11	14	17	20	22	25
58	.2042	.2070	.2098	.2127	.2155	.2184	.2212	31	3	6	9	11	14	17	20	23	25
59	.2212	.2241	.2270	.2299	.2327	.2356	.2386	30	3	6	9	12	15	17	20	23	26
60	0.2386	.2415	.2444	.2474	.2503	.2533	.2562	29	3	6	9	12	15	18	21	23	26
61	.2562	.2592	.2622	.2652	.2683	.2713	.2743	28	3	6	9	12	15	18	21	24	27
62	.2743	.2774	.2804	.2835	.2866	.2897	.2928	27	3	6	9	12	15	19	22	25	28
63	.2928	.2960	.2991	.3023	.3054	.3086	.3118	26	3	6	10	13	16	19	22	25	29
64	.3118	.3150	.3183	.3215	.3248	.3280	.3313	25	3	7	10	13	16	20	23	26	29
65	0.3313	.3346	.3380	.3413	.3447	.3480	.3514	24	3	7	10	13	17	20	23	27	30
66	.3514	.3548	.3583	.3617	.3652	.3686	.3721	23	3	7	10	14	17	21	24	28	31
67	.3721	.3757	.3792	.3828	.3864	.3900	.3936	22	4	7	11	14	18	22	25	29	32
68	.3936	.3972	.4009	.4046	.4083	.4121	.4158	21	4	7	11	15	19	22	26	30	33
69	.4158	.4196	.4234	.4273	.4311	.4350	.4389	20	4	8	12	15	19	23	27	31	35
70	0.4389	.4429	.4469	.4509	.4549	.4589	.4630	19	4	8	12	16	20	24	28	32	36
71	.4630	.4671	.4713	.4755	.4797	.4839	.4882	18	4	8	13	17	21	25	29	34	38
72	.4882	.4925	.4969	.5013	.5057	.5102	.5147	17	4	9	13	18	22	26	31	35	40
73	.5147	.5192	.5238	.5284	.5331	.5378	.5425	16	5	9	14	19	23	28	32	37	42
74	.5425	.5473	.5521	.5570	.5619	.5669	.5719	15	5	10	15	20	25	29	34	39	44
75	0.5719	.5770	.5822	.5873	.5926	.5979	.6032	14	5	10	16	21	26	31	37	42	47
76	.6032	.6086	.6141	.6196	.6252	.6309	.6366	13	6	11	17	22	28	33	39	45	50
77	.6366	.6424	.6483	.6542	.6603	.6664	.6725	12	6	12	18	24	30	36	42	48	54
78	.6725	.6788	.6851	.6915	.6981	.7047	.7114	11	7	13	20	26	33	39	46	52	59
79	.7113	.7181	.7250	.7320	.7391	.7464	.7537	10	7	14	21	28	35	42	49	57	64
80	0.7537	.7611	.7687	.7764	.7842	.7922	.8003	9				74–81					
81	.8003	.8085	.8169	.8255	.8342	.8431	.8522	8				82–91					
82	.8522	.8615	.8709	.8806	.8904	.9005	.9109	7				93–104					
83	.9109	.9214	.9322	.9433	.9547	.9664	.9784	6				105–120					
84	.9784	.9907	*.0034*	*.0164*	*.0299*	*.0437*	*.0580*	5	Differences			123–143					
85	1.0580	.0728	.0882	.1040	.1205	.1376	.1554	4				148–178					
86	.1554	.1739	.1933	.2135	.2348	.2571	.2806	3				185–235					
87	.2806	.3055	.3318	.3599	.3899	.4221	.4569	2				249–348					
88	.4569	.4947	.5362	.5819	.6331	.6911	.7581	1				378–670					
89	.7581	.8373	.9342	2.0591	.2352	.5363	∞	0				792–3011					
°		50′	40′	30′	20′	10′	0′	°	1	2	3	4	5	6	7	8	9

LOGARITHMIC COTANGENTS 0°–45° | PROPORTIONAL PARTS (Subtract)

	LOGARITHMS										PROPORTIONAL PARTS								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

	LOGARITHMS										PROPORTIONAL PARTS								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

INDEX

PAGE

Abbe prism 87
Abbe refractometer 87
Abbe theory 3
Aberrations 3, 28
  chromatic 28, 30, 35
  correction of 31
  spherical 28, 34
Adjustment
  of Babinet compensator . . . 217
  of Fresnel bi-prism 100
  of Fresnel mirrors 103
  of grating 62
  of interferometer . . . 121, 135
  of prism 77, 89
  of spectrometer 207
  of telescope and scale 216
Alpha Orionis 50
Alumina cream 168
Analyzer 139, 142
Angle 67, 84, 141
  critical 84
  polarizing 141
  prism 67
Angular magnification 8
Arc, mercury 198
Astigmatism 28, 29, 34
Axis, optic 146

Babinet compensator . . . 153, 217
Balance, analytical 215
Bartholinus 144
Base of prism 67
Bi-prism, Fresnel 97, 99
Brace prism 191
Brace-Lemon spectrophotometer 191
Brashear's process of silvering . 202
Brewster 141
Brewster's law 141, 185

PAGE

Cabinet
  for silvering 206
  for sugar preparations 169
Cane sugar 168
Cardinal points 2
Cathode deposit 205
Cauchy dispersion equation . 71, 75
Chromatic aberration . . . 28, 30, 35
Clarification of sugar solutions . 168
Cleaning glass surfaces 201
Collinear relation 3
Coma 28, 30
Combination of two systems . . 12
Conjugate foci 2, 32
Convergence ratio 8
Critical angle of refraction . . 84
Crystals 144
  biaxial 160
  colors in 155
  cut perpendicular to axis . . 158
  of Iceland spar 144
  positive and negative 145
  thin 155
  uniaxial 158
Curvature of field 28, 30, 35

Deviation, minimum 57, 68
  in grating 57
  in prism 68
Dextrose 166
Diffraction 36
Diffraction grating 53
Dioptre 14
Dispersion 59, 75, 161
  equation 71, 75
  of grating 59, 61
  of prism 70
  rotatory 161, 164

PAGE

Distortion 28, 30, 35
Double refraction 143
  by strain 157
  theory of 147
  in thin crystals 155

Edge of prism 67
Elliptical polarization . . . 153, 174
Extraordinary ray 144, 147
Eyepiece, Gauss 79, 209

Fabry-Perot interferometer . . 129
Focal lengths 2, 13
  measurement of 32
Foci, conjugate 2, 32
Foci, principal 2
  of combination 13
  of spherical refracting surface 24
Fresnel bi-prism 97, 99
Fresnel mirrors 94, 103
Fresnel's equations . . . . 183 ff.
Fringes, interference
      44, 91, 114, 125

Gauss eyepiece 79, 209
Gauss points 2
Geometrical optics 1
Grating, diffraction 44
Grating spectra 53
Grazing incidence . . . . . 85 ff.

Half-wave plate 148, 151
Huygens 144

Illumination . . . . 197 ff.
Image space 3
Images 2
Incidence, plane of 140
Index of refraction. *See* Refraction, index of
Intensity. *See* Maxima of Intensity *and* Minima of Intensity
Interference . . . . 44, 91 ff.
  conditions for 91
  methods of producing 94

PAGE

Interferometer . . . . 107 ff.
  Fabry-Perot 129
  Michelson 108
  parallel-plate 129
  stellar 49
Inversion of sugar 166

Lateral magnification. *See* Magnification
Laurent polarimeter 170
Lenses 2, 24
  testing of 33
  thin 2, 14, 16, 26
Levulose 166
Light rays 2
Limit of resolution. *See* Power, resolving

Magnification
  angular 8
  lateral 6, 23
  of spherical surface 23
Maxima of intensity, 39, 45, 56 ff., 133 f.
  for double slit 45
  for Fabry-Perot interferometer 133
    relation of, to minima . . . 134
    width of 133
  for *N* slits . . . . 56 ff.
    principal 56
      width of 58
    subsidiary 57
  for single slit 39
Mercury arc 198
Michelson interferometer 36, 49, 108
Micrometer calipers 212
Micrometer microscope 214
Minima of intensity, 39, 45, 56 ff., 133 f.
  for double slit 45
  for Fabry-Perot interferometer 133
    relation of, to maxima . . 134
  for *N* slits . . . . 56 ff.
  for single slit 39
Minimum deviation. *See* Deviation
Mirror 2
Mirrors, Fresnel 94, 103
Mount Wilson 49, 50

PAGE

N slits 53
Nicol prism 143, 149
Nodal points. *See* Points
Nodal slide 18, 19
Normal spectra 59

Object 2
Object space 3
Optic axis 146
Optical activity 161, 163
Optics, geometrical 1
Optics, physical 1
Ordinary ray 144, 147
Orionis, alpha 50
Oxy-gas flame 42, 199

Parallel-plate interferometer . . 129
  adjustment of 135
Physical optics 1
Plane 6, 7, 23, 140
  of incidence 140
  principal 6, 7
    of spherical surface 23
  unit 6, 7
Plate
  half-wave 148, 151
  parallel 129
  quarter-wave 152, 178
Points
  nodal 2, 8, 9
    experimental location of . . 9
    of spherical surface 23
  unit 6
Polarimeter 169
  Laurent 170
Polarization
  elliptical 174
  plane . . . . . 138, 142, 144 ff.
    by double refraction . . 144 ff.
    by reflection 138
    by transmission 142
  plane of 143
    rotation of . . . 160, 163, 186
Polarized light . . . . . . 138 ff.
  reflection of . . . . . . 182 ff.
Polarizer 139

PAGE

Power, resolving
  of double slit 48
  of grating 60
  measurement of 82
  of prism 73, 74, 82
  of single slit 41
Power of an optical system . . . 14
Principal focal length 13
  measurement of 32
Principal foci 5
  of combination 13
  of spherical surface 24
Principal planes and points . . 6, 7
Principal section 67

Quarter-wave plate 152
  use in elliptical polarization . 178

Ratio, convergence 8
Ray-tracing 22
Rectilinear propagation 1
Reflection 85, 182
  of polarized light 182
  total 85
Refracting angle 67, 84
  critical 84
  of prism 67
Refraction, double 143
  by strain 157
  theory of 147
  in thin crystals 152
Refraction, index of 28, 33
  by critical angle . . . . . 84 ff.
  in crystals 148
  by interferometer . . . 116, 125
  by prism 70
Refraction at spherical surface . 21
Refractometer 87
Resolving power. *See* Power, resolving
Rotation
  of plane of polarization
    160, 163, 186
  specific 164
    of cane sugar 172
Rotatory dispersion . . . 161, 164

PAGE
Saccharimeter 169
Scale 21, 211
  telescope and 21
  vernier 211
Silvering . . . . 202 ff.
  Brashear's process 202
  cabinet for 207
  cathode-deposit process . . . 205
Single-slit diffraction pattern . 36
Slide, nodal 18
Slit 36, 44
  double 44
  single 36
Slits, *N* 53
Specific rotation 164, 172
Spectra
  grating 45, 53
  normal 59
  overlapping 60
  prism 67
Spectrometer 76, 207
  adjustment of 207
  direct-vision 76
Spectrophotometer . . . . . . 190 ff.
  Brace 191
  Brace-Lemon 191
    adjustment of 193
Spectroscope. *See* Spectrometer
Spectroscopy, interferometer in . 117
Spherical aberration 28, 34

PAGE
Spherical surface, refraction at . 21
Spherometer 213
Stationary waves 93
Stellar interferometer 49
Sugar . . . . 165 ff.
  cabinet for testing 169
  cane 168
  invert 166
  testing of 165, 173

Telescope 15
  and scale 216
Testing 33, 165, 173
  of lenses 33
  of sugar 165, 173
Thickness of prism 72
Thin lenses. *See* Lenses
Total reflection 85
Two systems, combination of . . 12

Unit points 6

Vernier scale 211
Visibility curves 118

Wave-length tables 199
Waves, stationary 93
Wilson, Mount 49, 50
Wollaston prism 149

18643